Problem 5.19

(a) Find the density ρ of mobile charges in a piece of copper, assuming each atom contributes one free electron. [Look up the necessary physical constants.]

(b) Calculate the average electron velocity in a copper wire 1 mm in diameter, carrying a current of 1 A. [*Note*: this is literally a *snail's* pace. How, then, can you carry on a long distance telephone conversation?]

(c) What is the force of attraction between two such wires, 1 cm apart?

(d) If you could somehow remove the stationary positive ions, what would the electrical repulsion force be? How many times greater than the magnetic force is it?

Problem 5.20 Is Ampère's law consistent with the general rule (Eq. 1.46) that divergence-of-curl is always zero? Show that Ampère's law *cannot* be valid, in general, outside magnetostatics. Is there any such "defect" in the other three Maxwell equations?

Problem 5.21 Suppose there *did* exist magnetic monopoles. How would you modify Maxwell's equations and the force law, to accommodate them? If you think there are several plausible options, list them, and suggest how you might decide experimentally which one is right.

5.4 Magnetic Vector Potential

5.4.1 The Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{5.59}$$

The former is authorized by Theorem 1 (of Sect. 1.6.2), the latter by Theorem 2 (the proof of Theorem 2 is developed in Prob. 5.30). The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is *always* zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{5.60}$$

Now, the electric potential had a built-in ambiguity: you can add to V any function whose gradient is zero (which is to say, any *constant*), without altering the *physical* quantity $\mathbf{E}$. Likewise, you can add to the magnetic potential any function whose *curl* vanishes (which is to say, the *gradient of any scalar*), with no effect on $\mathbf{B}$. We can exploit this freedom to eliminate the divergence of $\mathbf{A}$:

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \tag{5.61}$$

To prove that this is always possible, suppose that our original potential, $\mathbf{A}_o$, is *not* divergenceless. If we add to it the gradient of λ ($\mathbf{A} = \mathbf{A}_o + \nabla\lambda$), the new divergence is

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_o + \nabla^2\lambda.$$

We can accommodate Eq. 5.61, then, if a function λ can be found that satisfies

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A}_o.$$

But this is *mathematically* identical to Poisson's equation (2.24),

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

with $\nabla \cdot \mathbf{A}_o$ in place of ρ/ϵ_0 as the "source." And we *know* how to solve Poisson's equation—that's what electrostatics is all about ("given the charge distribution, find the potential"). In particular, if ρ goes to zero at infinity, the solution is Eq. 2.29:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau',$$

and by the same token, if $\nabla \cdot \mathbf{A}_o$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_o}{\mathscr{r}}\, d\tau'.$$

If $\nabla \cdot \mathbf{A}_o$ does *not* go to zero at infinity, we'll have to use other means to discover the appropriate λ, just as we get the electric potential by other means when the charge distribution extends to infinity. But the *essential* point remains: *It is always possible to make the vector potential divergenceless.* To put it the other way around: The definition $\mathbf{B} = \nabla \times \mathbf{A}$ specifies the *curl* of $\mathbf{A}$, but it doesn't say anything about the *divergence*—we are at liberty to pick that as we see fit, and zero is ordinarily the simplest choice.

With this condition on $\mathbf{A}$, Ampère's law (5.60) becomes

$$\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}.} \tag{5.62}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[13] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.} \tag{5.63}$$

[13]In Cartesian coordinates, $\nabla^2\mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$, so Eq. 5.62 reduces to $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, and $\nabla^2 A_z = -\mu_0 J_z$. In curvilinear coordinates the unit vectors *themselves* are functions of position, and must be differentiated, so it is *not* the case, for example, that $\nabla^2 A_r = -\mu_0 J_r$. The safest way to calculate the Laplacian of a *vector*, in terms of its curvilinear components, is to use $\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A})$. Remember also that even if you *calculate* integrals such as 5.63 using curvilinear coordinates, you must first express $\mathbf{J}$ in terms of its *Cartesian* components (see Sect. 1.4.1).

For line and surface currents,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\,d\mathbf{l}'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\,da'. \tag{5.64}$$

(If the current does *not* go to zero at infinity, we have to find other ways to get **A**; some of these are explored in Ex. 5.12 and in the problems at the end of the section.)

It must be said that **A** is not as *useful* as V. For one thing, it's still a *vector*, and although Eqs. 5.63 and 5.64 are somewhat easier to work with than the Biot-Savart law, you still have to fuss with components. It would be nice if we could get away with a *scalar* potential,

$$\mathbf{B} = -\nabla U, \tag{5.65}$$

but this is incompatible with Ampère's law, since the curl of a gradient is always zero. (A **magnetostatic scalar potential** *can* be used, if you stick scrupulously to simply-connected, current-free regions, but as a theoretical tool it is of limited interest. See Prob. 5.28.) Moreover, since magnetic forces do no work, **A** does not admit a simple physical interpretation in terms of potential energy per unit charge. (In some contexts it can be interpreted as *momentum* per unit charge.[14]) Nevertheless, the vector potential has substantial theoretical importance, as we shall see in Chapter 10.

Example 5.11

A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$ (Fig. 5.45).

Solution: It might seem natural to align the polar axis along $\boldsymbol{\omega}$, but in fact the integration is easier if we let $\mathbf{r}$ lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle ψ. We may as well orient the x axis so that $\boldsymbol{\omega}$ lies in the xz plane, as shown in Fig. 5.46. According to Eq. 5.64,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da',$$

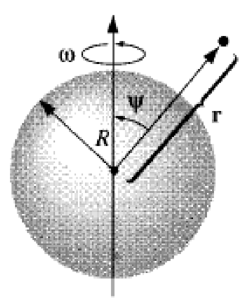

Figure 5.45

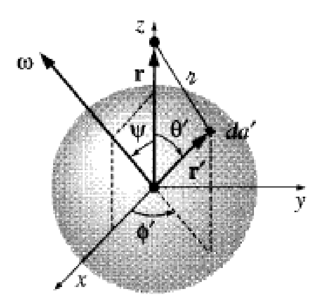

Figure 5.46

[14] M. D. Semon and J. R. Taylor, *Am. J. Phys.* **64**, 1361 (1996).

where $\mathbf{K} = \sigma \mathbf{v}$, $\imath = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2 \sin\theta'\, d\theta'\, d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega} \times \mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}} + (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi} \sin\phi'\, d\phi' = \int_0^{2\pi} \cos\phi'\, d\phi' = 0,$$

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2}\left(\int_0^{\pi} \frac{\cos\theta'\sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\, d\theta'\right)\hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}}\, du = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2}\sqrt{R^2 + r^2 - 2Rru}\,\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2}\left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r)\right].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\,\hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere}, \\[2ex] \dfrac{\mu_0 R^4 \sigma}{3r^3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere}. \end{cases} \tag{5.66}$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r \le R), \\[2ex] \dfrac{\mu_0 R^4 \omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases} \tag{5.67}$$

Curiously, the field inside this spherical shell is *uniform*:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}. \tag{5.68}$$

Example 5.12

Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Solution: This time we cannot use Eq. 5.64, since the current itself extends to infinity. But here's a cute method that does the job. Notice that

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi, \tag{5.69}$$

where Φ is the flux of $\mathbf{B}$ through the loop in question. This is reminiscent of Ampère's law in the integral form (5.55),

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

In fact, it's the same equation, with $\mathbf{B} \to \mathbf{A}$ and $\mu_0 I_{\text{enc}} \to \Phi$. If symmetry permits, we can determine $\mathbf{A}$ from Φ in the same way we got $\mathbf{B}$ from I_{enc}, in Sect. 5.3.3. The present problem (with a uniform longitudinal magnetic field $\mu_0 n I$ inside the solenoid and no field outside) is analogous to the Ampère's law problem of a fat wire carrying a uniformly distributed current. The vector potential is "circumferential" (it mimics the magnetic field of the wire); using a circular "amperian loop" at radius s *inside* the solenoid, we have

$$\oint \mathbf{A} \cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I(\pi s^2),$$

so

$$\mathbf{A} = \frac{\mu_0 n I}{2} s\,\hat{\boldsymbol{\phi}}, \quad \text{for } s < R. \tag{5.70}$$

For an amperian loop *outside* the solenoid, the flux is

$$\int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I(\pi R^2),$$

since the field only extends out to R. Thus

$$\mathbf{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, \quad \text{for } s > R. \tag{5.71}$$

If you have any doubts about this answer, *check* it: Does $\nabla \times \mathbf{A} = \mathbf{B}$? Does $\nabla \cdot \mathbf{A} = 0$? If so, we're done.

Typically, the direction of $\mathbf{A}$ will mimic the direction of the current. For instance, both were azimuthal in Exs. 5.11 and 5.12. Indeed, if all the current flows in *one* direction, then Eq. 5.63 suggests that $\mathbf{A}$ *must* point that way too. Thus the potential of a finite segment of straight wire (Prob. 5.22) is in the direction of the current. Of course, if the current extends to infinity you can't use Eq. 5.63 in the first place (see Probs. 5.25 and 5.26). Moreover, you can always add an arbitrary constant vector to $\mathbf{A}$—this is analogous to changing the reference point for V, and it won't affect the divergence or curl of $\mathbf{A}$, which is all that matters (in Eq. 5.63 we have chosen the constant so that $\mathbf{A}$ goes to zero at infinity). In principle you could even use a vector potential that is not divergenceless, in which case all bets are off. Despite all these caveats, the essential point remains: *Ordinarily* the direction of $\mathbf{A}$ will match the direction of the current.

Problem 5.22 Find the magnetic vector potential of a finite segment of straight wire, carrying a current I. [Put the wire on the z axis, from z_1 to z_2, and use Eq. 5.64.] Check that your answer is consistent with Eq. 5.35.

Problem 5.23 What current density would produce the vector potential, $\mathbf{A} = k\hat{\boldsymbol{\phi}}$ (where k is a constant), in cylindrical coordinates?

Problem 5.24 If $\mathbf{B}$ is *uniform*, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$ works. That is, check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?

Problem 5.25

(a) By whatever means you can think of (short of looking it up), find the vector potential a distance s from an infinite straight wire carrying a current I. Check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

(b) Find the magnetic potential *inside* the wire, if it has radius R and the current is uniformly distributed.

Problem 5.26 Find the vector potential above and below the plane surface current in Ex. 5.8.

Problem 5.27

(a) Check that Eq. 5.63 is consistent with Eq. 5.61, by applying the *divergence*.

(b) Check that Eq. 5.63 is consistent with Eq. 5.45, by applying the *curl*.

(c) Check that Eq. 5.63 is consistent with Eq. 5.62, by applying the *Laplacian*.

Problem 5.28 Suppose you want to define a magnetic scalar potential U (Eq. 5.65), in the vicinity of a current-carrying wire. First of all, you must stay away from the wire itself (there $\nabla \times \mathbf{B} \neq 0$); but that's not enough. Show, by applying Ampère's law to a path that starts at **a** and circles the wire, returning to **b** (Fig. 5.47), that the scalar potential cannot be single-valued (that is, $U(\mathbf{a}) \neq U(\mathbf{b})$, even if they represent the same physical point). As an example, find the scalar potential for an infinite straight wire. (To avoid a multivalued potential, you must restrict yourself to simply-connected regions that remain on one side or the other of every wire, never allowing you to go all the way around.)

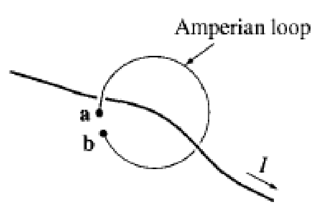

Figure 5.47

Problem 5.29 Use the results of Ex. 5.11 to find the field inside a uniformly charged sphere, of total charge Q and radius R, which is rotating at a constant angular velocity $\boldsymbol{\omega}$.

Problem 5.30

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field $\mathbf{F}$ can be written as the curl of a vector potential $\mathbf{A}$. What you have to do is find A_x, A_y, and A_z such that: (i) $\partial A_z/\partial y - \partial A_y/\partial z = F_x$; (ii) $\partial A_x/\partial z - \partial A_z/\partial x = F_y$; and (iii) $\partial A_y/\partial x - \partial A_x/\partial y = F_z$. Here's one way to do it: Pick $A_x = 0$, and solve (ii) and (iii) for A_y and A_z. Note that the "constants of integration" here are themselves functions of y and z—they're constant only with respect to x. Now plug these expressions into (i), and use the fact that $\nabla \cdot \mathbf{F} = 0$ to obtain

$$A_y = \int_0^x F_z(x', y, z)\, dx'; \quad A_z = \int_0^y F_x(0, y', z)\, dy' - \int_0^x F_y(x', y, z)\, dx'.$$

(b) By direct differentiation, check that the $\mathbf{A}$ you obtained in part (a) satisfies $\nabla \times \mathbf{A} = \mathbf{F}$. Is $\mathbf{A}$ divergenceless? [This was a very asymmetrical construction, and it would be surprising if it *were*—although we know that there *exists* a vector whose curl is $\mathbf{F}$ *and* whose divergence is zero.]

(c) As an example, let $\mathbf{F} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}}$. Calculate $\mathbf{A}$, and confirm that $\nabla \times \mathbf{A} = \mathbf{F}$. (For further discussion see Prob. 5.51.)

5.4.2 Summary; Magnetostatic Boundary Conditions

In Chapter 2, I drew a triangular diagram to summarize the relations among the three fundamental quantities of electrostatics: the charge density ρ, the electric field $\mathbf{E}$, and the potential V. A similar diagram can be constructed for magnetostatics (Fig. 5.48), relating

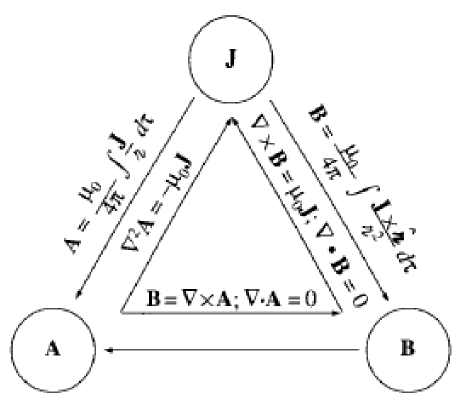

Figure 5.48