Temperature oscillations in a metal: Probing aspects of Fourier analysis

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The purpose of this experiment is to acquaint you with physical illustration of concepts in Fourier analysis all using a simple experimental setup involving wave-like behavior. In addition, you will measure the speed of propagation of thermal oscillations, analyze the heat equation and measure the thermal diffusivity of the material under observation. The temperature oscillations are distinct from true "travelling waves" as they do not transfer energy, and arise out of a diffusive or scattering process. But the concept of a "wave", as simple, innocuous and ubiquitous it seems, is exceedingly difficult and multi-faceted to the extent that no universal definition is possible [1]!

KEYWORDS

Heat equation · Fourier series · Fourier transform · Harmonics · Damping · Diffusivity.

APPROXIMATE PERFORMANCE TIME 1 week.

1 Objectives

In this experiment, we will,

1. understand the basis of heat flow and recognize heat conduction as a diffusive process,
2. learn about solutions of the heat equation,
3. decompose an oscillation into its harmonics,
4. observe different harmonics and how they damp with different rates, and
5. estimate the thermal diffusivity of a metal.

References

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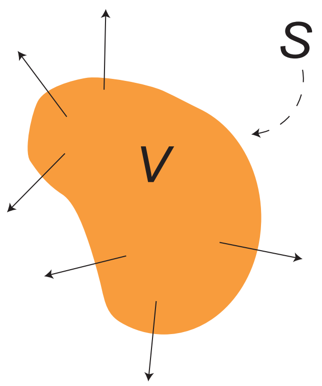
2 Theoretical background

2.1 Heat equation

Experiments have shown that heat flow is proportional to the gradient of the temperature. If the heat flux is $\vec{J}$, then

$$\vec{J} = -\kappa \vec{\nabla} T, \tag{1}$$



where $T(x, y, z, t)$ is the temperature, t is time and κ is defined as the thermal conductivity.

The heat flowing out of a volume, V, bounded by S, is,

$$\int_S \vec{J} \cdot \hat{n} dS = \int_V \vec{\nabla} \cdot \vec{J} dV. \tag{2}$$

Now the total thermal energy *inside* V is,

$$\int_V CT\,dV = \int_V \sigma\rho T\,dV, \tag{3}$$

with C being the specific heat capacity ($JK^{-1}mol^{-1}$), σ the per unit mass heat capacity ($JK^{-1}Kg^{-1}$) and ρ the mass density.

The rate of loss of energy through S is,

$$-\frac{\partial}{\partial t}\int_V \sigma\rho T\,dV = -\sigma\rho\int_V \frac{\partial T}{\partial t}dV. \tag{4}$$

Equating (2) and (4) we obtain,

$$\vec{\nabla}\cdot\vec{J} = -\sigma\rho\frac{\partial T}{\partial t}, \tag{5}$$

and substituting (1) into (5),

$$\vec{\nabla}^2 T = \frac{\sigma\rho}{\kappa}\frac{\partial T}{\partial t} \tag{6}$$

$$= \frac{1}{D}\frac{\partial T}{\partial t}, \tag{7}$$

where $D = \kappa/\sigma\rho$ is called the diffusivity of the material.

Q 1. How does the heat equation compare with (a) the wave equation $\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$, (b) the time dependent Schrodinger Equation?

Q 2. Show that for heat flow along a one-dimensional wire or rod, the heat equation becomes,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{D}\frac{\partial T}{\partial t}. \tag{8}$$

Q 3. Notice the analogies in the following table.

Conduction of heat	**Diffusion of particles**	**Electric current**
$\vec{J} = -\kappa\,\vec{\nabla}\,T$	$\vec{\Phi} = -D\,\vec{\nabla}\,n$	$\vec{J} = -\sigma\,\vec{\nabla}\,\phi$
κ: thermal conductivity	D: diffusion constant	σ: electrical conductivity
T: temperature	n: concentration of particles	ϕ: electric potential
$\vec{J}$: heat flow rate	$\vec{\Phi}$: flow rate of particles	$\vec{J}$: current density

Table 1: The analogies between apparently different physical phenomena, all unified through the heat equation.

Q 4. What are the SI Units of D and κ ?

2.2 Solving the heat equation

For the solution of one dimensional heat equation (Equation (8)) we can use the technique of separation of variables, and assume a solution of the form [4],

$$T(x,t) \propto \exp\big(i(kx - \omega t)\big). \tag{9}$$

Q 5. Show that a legitimate solution of the heat equation in the regime $x \geqslant 0$ is,

$$\begin{aligned} T(x,t) &= \sum_{\omega} \mathcal{F}(\omega) \exp(-i\omega t) \exp\left((i-1)\sqrt{\frac{\omega}{2D}}\, x\right) \\ &= \sum_{\omega} \mathcal{F}(\omega) \exp(-\sqrt{\frac{\omega}{2D}}\, x) \exp\big(i(\sqrt{\frac{\omega}{2D}}\, x - \omega t)\big). \end{aligned} \tag{10}$$

2.3 Fourier series of a square wave

In the present experiment we apply a periodic square pulse to a heater attached at one end of copper (Cu) rod.

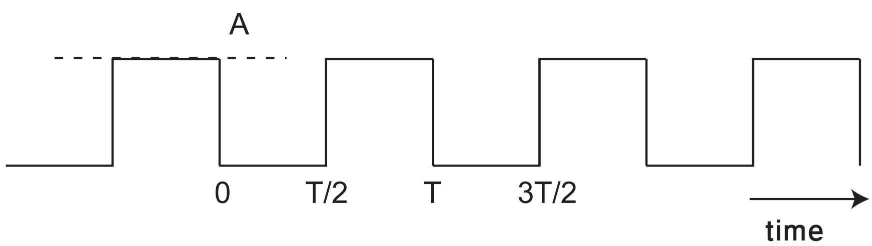


Figure 1: Sketch of a square pulse.

Q 6. Show that the Fourier series of a square wave, as shown in Figure 1, is given by,

$$\mathfrak{f}(t) = \frac{A}{2} - \frac{A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \frac{\sin(5\omega_0 t)}{5} + \dots\right), \tag{11}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency of the wave and A is its amplitude.

Q 7. What is the average value of the square wave?

Q 8. Observe the presence of only the odd harmonics in $\mathfrak{f}(t)$. Can you verify the frequency components presented in a square wave using the concept of Fourier transform? Plot the Fourier transform of a simulated square pulse in Matlab.

2.4 Applying boundary conditions

A particular solution to Equation (10) can be found if we apply the appropriate boundary conditions. In our case these conditions are determined by the square pulse heating waveform applied to the end of the rod, arbitrarily set at $x = 0$. Therefore,

$$T(0,t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)\omega_0 t) \tag{12}$$

where $n = 0, 1, 2, 3\ldots$ and A represents the maximum temperature. Furthermore, after substituting $x = 0$ into Equation (10), we have,

$$T(0,t) = \sum_{\omega}\mathcal{F}(\omega)\exp(-i\omega t). \tag{13}$$

Q 9. Identify the Fourier relationship between $T(0,t)$ and $\mathcal{F}(\omega)$ in Equation (13)?

Equations (12) and (13) represent the same pulsing waveform and are necessarily equal. Hence, expressing (12) in complex form,

$$T(0,t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=0}^{\infty}\left(\frac{e^{i(2n+1)\omega_0 t} - e^{-i(2n+1)\omega_0 t}}{2i(2n+1)}\right). \tag{14}$$

Comparing Equations (13) and (14), the only nonzero values that occur for $A(\omega)$ are,

$$\mathcal{F}(0) = \frac{A}{2} \tag{15}$$

$$\mathcal{F}((2n+1)\omega_0) = \frac{-iA}{2\pi(2n+1)} \tag{16}$$

$$\mathcal{F}(-(2n+1)\omega_0) = \frac{iA}{2\pi(2n+1)}. \tag{17}$$

Putting these conditions back into (10), we derive the following particular solution,

$$T(x,t) = \frac{A}{2} - \sum_{n=0}^{\infty}\frac{iA}{(2n+1)2\pi}\exp\left(-\sqrt{\frac{(2n+1)\omega_0}{2D}}x\right)\exp\left(i(\sqrt{\frac{(2n+1)\omega_0}{2D}}x - (2n+1)\omega_0 t)\right)$$
$$+\sum_{n=0}^{\infty}\frac{iA}{(2n+1)2\pi}\exp\left(-\sqrt{\frac{-(2n+1)\omega_0}{2D}}x\right)\exp\left(i(\sqrt{\frac{-(2n+1)\omega_0}{2D}}x + (2n+1)\omega_0 t)\right).$$

After some fairly simple algebraic reshuffling,

$$T(x,t) = \frac{A}{2} - \sum_{n=0}^{\infty}\frac{iA}{(2n+1)2\pi}\exp\left(-\sqrt{\frac{(2n+1)\omega_0}{2D}}x\right)\exp\left(i(\sqrt{\frac{(2n+1)\omega_0}{2D}}x - (2n+1)\omega_0 t)\right) \tag{18}$$
$$+\sum_{n=0}^{\infty}\frac{iA}{(2n+1)2\pi}\exp\left(-\sqrt{\frac{(2n+1)\omega_0}{2D}}x\right)\exp\left(-i(\sqrt{\frac{(2n+1)\omega_0}{2D}}x - (2n+1)\omega_0 t)\right). \tag{19}$$

Making the identifications, $(2n+1)\omega_0 = \omega_{2n+1}$, $-(2n+1)\omega_0 = -\omega_{2n+1}$, as well as,

$$\mathcal{F}(0) = \frac{A}{2}$$

$$\mathcal{F}(\omega_{2n+1}) = \frac{-iA}{2\pi(2n+1)}$$

$$\mathcal{F}(-\omega_{2n+1}) = \frac{iA}{2\pi(2n+1)} = (A(\omega_{2n+1}))^* = A^*(\omega_{2n+1}),$$

the solution can be compactly written as,

$$T(x,t) = \frac{A}{2} + \sum_{n=0}^{\infty} \mathcal{F}(\omega_{2n+1}) \exp(-i\omega_{2n+1}t) \exp\left((i-1)\sqrt{\frac{\omega_{2n+1}}{2D}}x\right) \tag{20}$$

$$+ \sum_{n=0}^{\infty} \mathcal{F}(-\omega_{2n+1}) \exp(i\omega_{2n+1}t) \exp\left((-i-1)\sqrt{\frac{\omega_{2n+1}}{2D}}x\right)$$

$$= \frac{A}{2} + \sum_{n=0}^{\infty} \mathcal{F}(\omega_{2n+1}) \exp(-i\omega_{2n+1}t) \exp\left((i-1)\sqrt{\frac{\omega_{2n+1}}{2D}}x\right)$$

$$+ \sum_{n=0}^{\infty} \mathcal{F}^*(\omega_{2n+1}) \exp(i\omega_{2n+1}t) \exp\left((-i-1)\sqrt{\frac{\omega_{2n+1}}{2D}}x\right). \tag{21}$$

Since the second and third terms on R.H.S of Equation (21) are complex conjugates of each other, we can write,

$$T(x,t) = \frac{A}{2} + 2\sum_{n=0}^{\infty} \mathrm{Re}[\mathcal{F}(\omega_{2n+1}) \exp(-i\omega_{2n+1}t) \exp\left((i-1)\sqrt{\frac{\omega_{2n+1}}{2D}}x\right)] \tag{22}$$

$$= \frac{A}{2} + \frac{A}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \exp\left(-\sqrt{\frac{\omega_{2n+1}}{2D}}x\right) \sin\left(\sqrt{\frac{\omega_{2n+1}}{2D}}x - \omega_{2n+1}t\right)$$

Q 10. At a fixed x, time harmonics of which order are present in the temperature oscillations?

Q 11. Define a damping length $\delta_n = \sqrt{\frac{2D}{\omega_n}}$ for the n'th harmonic. How does δ_n vary with n?

The temperature oscillations shown in the solution (22) illustrate the superposition of damped oscillations.

Q 12. Calculate the ratio of the n'th harmonic to the fundamental frequency.

It can be seen that the higher harmonics damp out very quickly because the damping length increases with frequency. Therefore, at a sufficient distance from the origin, we can also approximate the temperature distribution through the first harmonic only.

2.4.1 An everyday example

The surface of the Earth is heated by a diurnal temperature cycle that can be approximated by the sinusoidal variation

$$T_0 + \Delta T \cos(\Omega t), \tag{23}$$

where $\Omega = 2\pi/24\ \mathrm{h}^{-1}$.

Q 13. How far into the Earth's surface do the temperature oscillations penetrate? The average thermal diffusivity D of the Earth's crust is $\sim 1\ \mathrm{mm^2 s^{-1}}$[6].

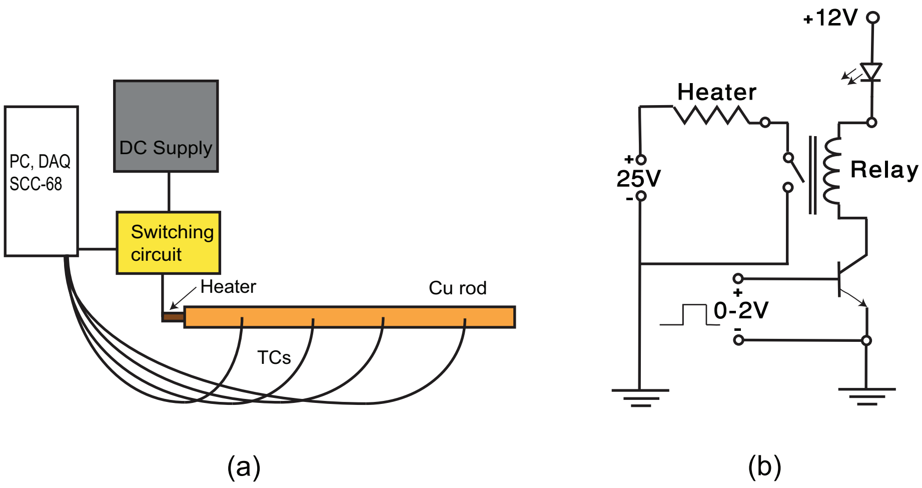


Figure 2: (a) is the schematic of the experimental setup and (b) shows the circuitry.

3 The experiment

Figure 2 shows the experimental setup. A copper rod of length approximately 0.5 m and diameter 30 mm with four thermocouples clamped to it, equidistantly and 3 cm apart, arranged along the rod. The metal bar is heated with a square pulse, at a rate of 5 mHz using a 25 W cartridge heater which is inserted into the rod after applying thermal wax. The heater is connected to a simple electric circuit shown in Figure (2). The relay is controlled using Labview, which sends a square pulse to the relay.

Q 14. Find out how a relay works and what is its purpose in the present experiment?

Note: The maximum voltage that could be applied to the heater is 25 volts. The rod is wrapped inside a flexible fiberglass, which acts as an insulator.

3.1 Scheme

K type thermocouples have been attached and the Labview code has already been prepared (It can be downloaded from the website of the course). You have to

1. check the cold junction compensation (CJC) value,
2. apply a square pulse to the heater,
3. acquire the thermocouple data,
4. save the data to a file, and
5. plot the data and be able to investigate the following points. You are required to come up with suitable graphs to illustrate and respond to the these points of inquiry.

Leave the system running for some time until the dynamic equilibrium state has been reached.

Q 15. How will you confirm that such a state is achieved?

Q 16. Plot all the thermocouple data in Matlab. What do you observe?

Q 17. Take the Fourier transform. What does the peak at zero frequency specify and how can it be removed? In Matlab, ones uses the command fftshift(fft(...)) to find the Fourier transform. Understand the units on the horizontal axis.

Q 18. Determine the damping length on each harmonic.

Q 19. Determine the 'velocity' of the 'wave'.

Q 20. Determine the thermal diffusivity based on the phase velocity.

Q 21. Determine the thermal diffusivity based on the damping coefficient. Compare your diffusivity value with the published values [3].

Q 22. Compare your results from the preceding two equations and comment on the accuracy, and the relative accord (or discord) between them.

Q 23. Plot the Fourier transform of your data.

Q 24. Do you observe the higher frequency data to damp out more quickly? Back up your observations with quantitative results.

Q 25. What is the skin depth of the temperature oscillations?

3.2 Comments on energy transfer

You must have noticed that we have refrained from freely labelling these temperature oscillations as 'thermal waves'. The first reason is that these oscillations are solutions of the heat equation, not the wave equation which involves a second time derivative. Heat conduction is a diffusive rather than a traveling wave. The second reason which endorses this point of view is that these oscillations do not transport energy [5].

Q 26. Using the temperature oscillations, Equation (22), determine the heat transfer rate $\vec{J} = -\kappa\vec{\nabla}T$ through the Cu rod. Show that the so called thermal 'waves' do not carry energy and hence, they are fundamentally different from sound waves or electromagnetic waves.

Q 27. Contrary to the theoretical suggestion, we know that as one end of the Cu rod is heated the end does get hot. How does one resolve this paradox?

Q 28. Based on the data you have acquired, calculate the thermal conductivity of Cu using Equation (1) and compare with your previously determined values.