A very special kind of motion occurs when the force acting on a body is proportional to the displacement of the body from some equilibrium position. If this force is always directed toward the equilibrium position, repetitive back-and-forth motion occurs about this position. Such motion is called periodic motion, harmonic motion, oscillation, or vibration (the four terms are completely equivalent).

You are most likely familiar with several examples of periodic motion, such as the oscillations of a block attached to a spring, the swinging of a child on a playground swing, the motion of a pendulum, and the vibrations of a stringed musical instrument. In addition to these everyday examples, numerous other systems exhibit periodic motion. For example, the molecules in a solid oscillate about their equilibrium positions; electromagnetic waves, such as light waves, radar, and radio waves, are characterized by oscillating electric and magnetic field vectors; and in alternating-current electrical circuits, voltage, current, and electrical charge vary periodically with time.

Most of the material in this chapter deals with simple harmonic motion, in which an object oscillates such that its position is specified by a sinusoidal function of time with no loss in mechanical energy. In real mechanical systems, damping (frictional) forces are often present. These forces are considered in optional Section 13.6 at the end of this chapter.

### 13.1 SIMPLE HARMONIC MOTION

Consider a physical system that consists of a block of mass $m$ attached to the end of a spring, with the block free to move on a horizontal, frictionless surface (Fig. 13.1). When the spring is neither stretched nor compressed, the block is at the position $x = 0$, called the equilibrium position of the system. We know from experience that such a system oscillates back and forth if disturbed from its equilibrium position.

We can understand the motion in Figure 13.1 qualitatively by first recalling that when the block is displaced a small distance $x$ from equilibrium, the spring exerts on the block a force that is proportional to the displacement and given by Hooke’s law (see Section 7.3):

$$ F_s = -kx $$

(13.1)

We call this a restoring force because it is always directed toward the equilibrium position and therefore opposite the displacement. That is, when the block is displaced to the right of $x = 0$ in Figure 13.1, then the displacement is positive and the restoring force is directed to the left. When the block is displaced to the left of $x = 0$, then the displacement is negative and the restoring force is directed to the right.

Applying Newton’s second law to the motion of the block, together with Equation 13.1, we obtain

$$ F_s = -kx = ma $$

$$ a = -\frac{k}{m}x $$

(13.2)

That is, the acceleration is proportional to the displacement of the block, and its direction is opposite the direction of the displacement. Systems that behave in this way are said to exhibit simple harmonic motion. An object moves with simple harmonic motion whenever its acceleration is proportional to its displacement from some equilibrium position and is oppositely directed.
An experimental arrangement that exhibits simple harmonic motion is illustrated in Figure 13.2. A mass oscillating vertically on a spring has a pen attached to it. While the mass is oscillating, a sheet of paper is moved perpendicular to the direction of motion of the spring, and the pen traces out a wavelike pattern.

In general, a particle moving along the $x$ axis exhibits simple harmonic motion when $x$, the particle’s displacement from equilibrium, varies in time according to the relationship

$$ x = A \cos(\omega t + \phi) $$

(13.3)

where $A$, $\omega$, and $\phi$ are constants. To give physical significance to these constants, we have labeled a plot of $x$ as a function of $t$ in Figure 13.3a. This is just the pattern that is observed with the experimental apparatus shown in Figure 13.2. The amplitude $A$ of the motion is the maximum displacement of the particle in either the positive or negative $x$ direction. The constant $\omega$ is called the angular frequency of the motion and has units of radians per second. (We shall discuss the geometric significance of $\omega$ in Section 13.2.) The constant angle $\phi$, called the phase constant (or phase angle), is determined by the initial displacement and velocity of the particle. If the particle is at its maximum position at $t = 0$, then $\phi = 0$ and the curve of $x$ versus $t$ is as shown in Figure 13.3b. If the particle is at some other position at $t = 0$, the constants $\phi$ and $A$ tell us what the position was at time $t = 0$. The quantity $(\omega t + \phi)$ is called the phase of the motion and is useful in comparing the motions of two oscillators.

Note from Equation 13.3 that the trigonometric function $x$ is periodic and repeats itself every time $\omega t$ increases by $2\pi$ rad. The period $T$ of the motion is the time it takes for the particle to go through one full cycle. We say that the particle has made one oscillation. This definition of $T$ tells us that the value of $x$ at time $t$ equals the value of $x$ at time $t + T$. We can show that $T = 2\pi/\omega$ by using the preceding observation that the phase $(\omega t + \phi)$ increases by $2\pi$ rad in a time $T$:

$$ \omega t + \phi + 2\pi = \omega(t + T) + \phi $$

Hence, $\omega T = 2\pi$, or

$$ T = \frac{2\pi}{\omega} $$

(13.4)
The inverse of the period is called the **frequency** $f$ of the motion. The frequency represents the number of oscillations that the particle makes per unit time:

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$  \hspace{1cm} (13.5)

The units of $f$ are cycles per second = s$^{-1}$, or **hertz** (Hz).

Rearranging Equation 13.5, we obtain the angular frequency:

$$\omega = 2\pi f = \frac{2\pi}{T}$$  \hspace{1cm} (13.6)

**Quick Quiz 13.1**

What would the phase constant $\phi$ have to be in Equation 13.3 if we were describing an oscillating object that happened to be at the origin at $t = 0$?

**Quick Quiz 13.2**

An object undergoes simple harmonic motion of amplitude $A$. Through what total distance does the object move during one complete cycle of its motion? (a) $A/2$. (b) $A$. (c) $2A$. (d) $4A$.

We can obtain the linear velocity of a particle undergoing simple harmonic motion by differentiating Equation 13.3 with respect to time:

$$v = \frac{dx}{dt} = -\omega A \sin(\alpha t + \phi)$$  \hspace{1cm} (13.7)

The acceleration of the particle is

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi)$$  \hspace{1cm} (13.8)

Because $x = A \cos(\omega t + \phi)$, we can express Equation 13.8 in the form

$$a = -\omega^2 x$$  \hspace{1cm} (13.9)

From Equation 13.7 we see that, because the sine function oscillates between $\pm 1$, the extreme values of $v$ are $\pm \omega A$. Because the cosine function also oscillates between $\pm 1$, Equation 13.8 tells us that the extreme values of $a$ are $\pm \omega^2 A$. Therefore, the maximum speed and the magnitude of the maximum acceleration of a particle moving in simple harmonic motion are

$$v_{\text{max}} = \omega A$$  \hspace{1cm} (13.10)

$$a_{\text{max}} = \omega^2 A$$  \hspace{1cm} (13.11)

Figure 13.4a represents the displacement versus time for an arbitrary value of the phase constant. The velocity and acceleration curves are illustrated in Figure 13.4b and c. These curves show that the phase of the velocity differs from the phase of the displacement by $\pi/2$ rad, or $90^\circ$. That is, when $x$ is a maximum or a minimum, the velocity is zero. Likewise, when $x$ is zero, the speed is a maximum.
Furthermore, note that the phase of the acceleration differs from the phase of the displacement by $\pi$ rad, or 180°. That is, when $x$ is a maximum, $a$ is a maximum in the opposite direction.

The phase constant $\phi$ is important when we compare the motion of two or more oscillating objects. Imagine two identical pendulum bobs swinging side by side in simple harmonic motion, with one having been released later than the other. The pendulum bobs have different phase constants. Let us show how the phase constant and the amplitude of any particle moving in simple harmonic motion can be determined if we know the particle’s initial speed and position and the angular frequency of its motion.

Suppose that at $t = 0$ the initial position of a single oscillator is $x = x_i$ and its initial speed is $v = v_i$. Under these conditions, Equations 13.3 and 13.7 give

$$x_i = A \cos \phi$$

$$v_i = -\omega A \sin \phi$$

Dividing Equation 13.13 by Equation 13.12 eliminates $A$, giving $v_i/x_i = -\omega \tan \phi$, or

$$\tan \phi = -\frac{v_i}{\omega x_i}$$

Furthermore, if we square Equations 13.12 and 13.13, divide the velocity equation by $\omega^2$, and then add terms, we obtain

$$x_i^2 + \left(\frac{v_i}{\omega}\right)^2 = A^2 \cos^2 \phi + A^2 \sin^2 \phi$$

Using the identity $\sin^2 \phi + \cos^2 \phi = 1$, we can solve for $A$:

$$A = \sqrt{x_i^2 + \left(\frac{v_i}{\omega}\right)^2}$$
The following properties of a particle moving in simple harmonic motion are important:

- The acceleration of the particle is proportional to the displacement but is in the opposite direction. This is the necessary and sufficient condition for simple harmonic motion, as opposed to all other kinds of vibration.
- The displacement from the equilibrium position, velocity, and acceleration all vary sinusoidally with time but are not in phase, as shown in Figure 13.4.
- The frequency and the period of the motion are independent of the amplitude. (We show this explicitly in the next section.)

Can we use Equations 2.8, 2.10, 2.11, and 2.12 (see pages 35 and 36) to describe the motion of a simple harmonic oscillator?

**Quick Quiz 13.3**

Can we use Equations 2.8, 2.10, 2.11, and 2.12 (see pages 35 and 36) to describe the motion of a simple harmonic oscillator?

**Example 13.1** An Oscillating Object

An object oscillates with simple harmonic motion along the x axis. Its displacement from the origin varies with time according to the equation

\[ x = (4.00 \text{ m}) \cos \left( \pi t + \frac{\pi}{4} \right) \]

where \( t \) is in seconds and the angles in the parentheses are in radians. (a) Determine the amplitude, frequency, and period of the motion.

**Solution** By comparing this equation with Equation 13.3, the general equation for simple harmonic motion—

\[ x = A \cos(\omega t + \phi) \]

—we see that \( A = 4.00 \text{ m} \) and \( \omega = \pi \text{ rad/s} \). Therefore, \( f = \omega/2\pi = \pi/2\pi = 0.500 \text{ Hz} \) and \( T = 1/f = 2.00 \text{ s} \).

(b) Calculate the velocity and acceleration of the object at any time.

**Solution**

\[ v = \frac{dx}{dt} = -(4.00 \text{ m/s}) \sin \left( \pi t + \frac{\pi}{4} \right) \frac{d}{dt} (\pi t) \]

\[ a = \frac{dv}{dt} = -(4.00 \text{ m/s}) \cos \left( \pi t + \frac{\pi}{4} \right) \frac{d}{dt} (\pi t) \]

(c) Using the results of part (b), determine the position, velocity, and acceleration of the object at \( t = 1.00 \text{ s} \).

**Solution** Noting that the angles in the trigonometric functions are in radians, we obtain, at \( t = 1.00 \text{ s} \),

\[ x = (4.00 \text{ m}) \cos \left( \pi + \frac{\pi}{4} \right) = (4.00 \text{ m}) \cos \left( \frac{5\pi}{4} \right) \]

\[ = (4.00 \text{ m})(-0.707) = -2.83 \text{ m} \]

\[ v = -(4.00\pi \text{ m/s}) \sin \left( \frac{5\pi}{4} \right) = -(4.00\pi \text{ m/s})(-0.707) \]

\[ = 8.89 \text{ m/s} \]

\[ a = -(4.00\pi^2 \text{ m/s}^2) \cos \left( \frac{5\pi}{4} \right) \]

\[ = -(4.00\pi^2 \text{ m/s}^2)(-0.707) = 27.9 \text{ m/s}^2 \]

(d) Determine the maximum speed and maximum acceleration of the object.

**Solution** In the general expressions for \( v \) and \( a \) found in part (b), we use the fact that the maximum values of the sine and cosine functions are unity. Therefore, \( v \) varies between \( \pm 4.00\pi \text{ m/s} \), and \( a \) varies between \( \pm 4.00\pi^2 \text{ m/s}^2 \). Thus,

\[ v_{\text{max}} = 4.00\pi \text{ m/s} = 12.6 \text{ m/s} \]

\[ a_{\text{max}} = 4.00\pi^2 \text{ m/s}^2 = 39.5 \text{ m/s}^2 \]

We obtain the same results using \( v_{\text{max}} = \omega A \) and \( a_{\text{max}} = \omega^2 A \), where \( A = 4.00 \text{ m} \) and \( \omega = \pi \text{ rad/s} \).

(e) Find the displacement of the object between \( t = 0 \) and \( t = 1.00 \text{ s} \).
## 13.2 The Block—Spring System Revisited

Let us return to the block—spring system (Fig. 13.5). Again we assume that the surface is frictionless; hence, when the block is displaced from equilibrium, the only force acting on it is the restoring force of the spring. As we saw in Equation 13.2, when the block is displaced a distance $x$ from equilibrium, it experiences an acceleration $a = -(k/m)x$. If the block is displaced a maximum distance $x = A$ at some initial time and then released from rest, its initial acceleration at that instant is $-kA/m$ (its extreme negative value). When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum. The block then continues to travel to the left of equilibrium and finally reaches $x = -A$, at which time its acceleration is $kA/m$ (maximum positive) and its speed is again zero. Thus, we see that the block oscillates between the turning points $x = \pm A$.

Let us now describe the oscillating motion in a quantitative fashion. Recall that $a = dv/dt = d^2x/dt^2$, and so we can express Equation 13.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

(13.16)

If we denote the ratio $k/m$ with the symbol $\omega^2$, this equation becomes

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

(13.17)

Now we require a solution to Equation 13.17— that is, a function $x(t)$ that satisfies this second-order differential equation. Because Equations 13.17 and 13.9 are equivalent, each solution must be that of simple harmonic motion:

$$x = A \cos(\omega t + \phi)$$

To see this explicitly, assume that $x = A \cos(\omega t + \phi)$. Then

$$\frac{dx}{dt} = A \frac{d}{dt} \cos(\omega t + \phi) = -\omega A \sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -\omega A \frac{d}{dt} \sin(\omega t + \phi) = -\omega^2 A \cos(\omega t + \phi)$$

Comparing the expressions for $x$ and $d^2x/dt^2$, we see that $d^2x/dt^2 = -\omega^2 x$, and Equation 13.17 is satisfied. We conclude that whenever the force acting on a particle is linearly proportional to the displacement from some equilibrium

---

### Solution

The $x$ coordinate at $t = 0$ is

$$x_f = (4.00 \text{ m}) \cos \left( 0 + \frac{\pi}{4} \right) = (4.00 \text{ m})(0.707) = 2.83 \text{ m}$$

In part (c), we found that the $x$ coordinate at $t = 1.00 \text{ s}$ is $-2.83 \text{ m}$; therefore, the displacement between $t = 0$ and $t = 1.00 \text{ s}$ is

$$\Delta x = x_f - x_i = -2.83 \text{ m} - 2.83 \text{ m} = -5.66 \text{ m}$$

Because the object’s velocity changes sign during the first second, the magnitude of $\Delta x$ is not the same as the distance traveled in the first second. (By the time the first second is over, the object has been through the point $x = -2.83 \text{ m}$ once, traveled to $x = -4.00 \text{ m}$, and come back to $x = -2.83 \text{ m}$.)

### Exercise

What is the phase of the motion at $t = 2.00 \text{ s}$?

**Answer** $9\pi/4 \text{ rad.}$
position and in the opposite direction \( F = -kx \), the particle moves in simple harmonic motion.

Recall that the period of any simple harmonic oscillator is \( T = 2\pi/\omega \) (Eq. 13.4) and that the frequency is the inverse of the period. We know from Equations 13.16 and 13.17 that \( \omega = \sqrt{k/m} \), so we can express the period and frequency of the block–spring system as

\[
T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{13.18}
\]

\[
f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{13.19}
\]

That is, the frequency and period depend only on the mass of the block and on the force constant of the spring. Furthermore, the frequency and period are independent of the amplitude of the motion. As we might expect, the frequency is greater for a stiffer spring (the stiffer the spring, the greater the value of \( k \)) and decreases with increasing mass.

**Special Case 1.** Let us consider a special case to better understand the physical significance of Equation 13.3, the defining expression for simple harmonic motion. We shall use this equation to describe the motion of an oscillating block–spring system. Suppose we pull the block a distance \( A \) from equilibrium and then release it from rest at this stretched position, as shown in Figure 13.6. Our solution for \( x \) must obey the initial conditions that \( x_i = A \) and \( v_i = 0 \) at \( t = 0 \). It does if we choose \( \phi = 0 \), which gives \( x = A \cos \omega t \) as the solution. To check this solution, we note that it satisfies the condition that \( x_i = A \) at \( t = 0 \) because \( \cos 0 = 1 \). Thus, we see that \( A \) and \( \phi \) contain the information on initial conditions.

Now let us investigate the behavior of the velocity and acceleration for this special case. Because \( x = A \cos \omega t \),

\[
v = \frac{dx}{dt} = -\omega A \sin \omega t
\]

\[
a = \frac{dv}{dt} = -\omega^2 A \cos \omega t
\]

From the velocity expression we see that, because \( \sin 0 = 0 \), \( v_i = 0 \) at \( t = 0 \), as we require. The expression for the acceleration tells us that \( a = -\omega^2 A \) at \( t = 0 \). Physically, this negative acceleration makes sense because the force acting on the block is directed to the left when the displacement is positive. In fact, at the extreme po-
13.2 The Block—Spring System Revisited

The Block—Spring System Revisited

Another approach to showing that \( \cos\omega t \) is the correct solution involves using the relationship \( \tan \omega t = \frac{\sin \omega t}{\cos \omega t} \) (Eq. 13.14). Because at \( \tan \omega t = 0 \) and thus \( \phi = 0 \), (The tangent of \( \pi \) also equals zero, but \( \phi = \pi \) gives the wrong value for \( x_i \).)

Figure 13.7 is a plot of displacement, velocity, and acceleration versus time for this special case. Note that the acceleration reaches extreme values of \( -2\omega^2A \) while the displacement has extreme values of \( -A \) because the force is maximal at those positions. Furthermore, the velocity has extreme values of \( \pm\omega A \), which both occur at \( x = 0 \). Hence, the quantitative solution agrees with our qualitative description of this system.

Special Case 2. Now suppose that the block is given an initial velocity \( v_i \) to the right at the instant it is at the equilibrium position, so that \( x_i = 0 \) and \( v = v_i \) at \( t = 0 \) (Fig. 13.8). The expression for \( x \) must now satisfy these initial conditions. Because the block is moving in the positive \( x \) direction at \( t = 0 \) and because \( x_i = 0 \) at \( t = 0 \), the expression for \( x \) must have the form \( x = A \sin \omega t \).

Applying Equation 13.14 and the initial condition that \( x_i = 0 \) at \( t = 0 \), we find that \( \tan \phi = -\infty \) and \( \phi = -\pi/2 \). Hence, Equation 13.3 becomes \( x = A \cos (\omega t - \pi/2) \), which can be written \( x = A \sin \omega t \). Furthermore, from Equation 13.15 we see that \( A = v_i/\omega \); therefore, we can express \( x \) as

\[
x = \frac{v_i}{\omega} \sin \omega t
\]

The velocity and acceleration in this case are

\[
v = \frac{dx}{dt} = v_i \cos \omega t
\]
\[
a = \frac{dv}{dt} = -\omega v_i \sin \omega t
\]

These results are consistent with the facts that (1) the block always has a maximum...
speed at \( x = 0 \) and (2) the force and acceleration are zero at this position. The graphs of these functions versus time in Figure 13.7 correspond to the origin at \( O' \).

**Quick Quiz 13.4**

What is the solution for \( x \) if the block is initially moving to the left in Figure 13.8?

**Example 13.2**  
Watch Out for Potholes!

A car with a mass of 1,300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20,000 N/m. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road.

**Solution**  
We assume that the mass is evenly distributed. Thus, each spring supports one fourth of the load. The total mass is 1,460 kg, and therefore each spring supports 365 kg.

Hence, the frequency of vibration is, from Equation 13.19,

\[
 f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{20,000 \text{ N/m}}{365 \text{ kg}}} = 1.18 \text{ Hz}
\]

**Exercise**  
How long does it take the car to execute two complete vibrations?

**Answer**  
1.70 s.

**Example 13.3**  
A Block–Spring System

A block with a mass of 200 g is connected to a light spring for which the force constant is 5.00 N/m and is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from rest, as shown in Figure 13.6. (a) Find the period of its motion.

**Solution**  
From Equations 13.16 and 13.17, we know that the angular frequency of any block–spring system is

\[
 \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.00 \text{ N/m}}{200 \times 10^{-3} \text{ kg}}} = 5.00 \text{ rad/s}
\]

and the period is

\[
 T = \frac{2\pi}{\omega} = \frac{2\pi}{5.00 \text{ rad/s}} = 1.26 \text{ s}
\]

(b) Determine the maximum speed of the block.

**Solution**  
We use Equation 13.10:

\[
 v_{\text{max}} = \omega A = (5.00 \text{ rad/s})(5.00 \times 10^{-2} \text{ m}) = 0.250 \text{ m/s}
\]

(c) What is the maximum acceleration of the block?

**Solution**  
We use Equation 13.11:

\[
 a_{\text{max}} = \omega^2 A = (5.00 \text{ rad/s})^2(5.00 \times 10^{-2} \text{ m}) = 1.25 \text{ m/s}^2
\]

(d) Express the displacement, speed, and acceleration as functions of time.

**Solution**  
This situation corresponds to Special Case 1, where our solution is \( x = A \cos \omega t \). Using this expression and the results from (a), (b), and (c), we find that

\[
 x = A \cos \omega t = (0.050 \text{ m}) \cos 5.00t
\]

\[
 v = \omega A \sin \omega t = -0.250 \text{ m/s} \sin 5.00t
\]

\[
 a = \omega^2 A \cos \omega t = -(1.25 \text{ m/s}^2) \cos 5.00t
\]

### 13.3 Energy of the Simple Harmonic Oscillator

Let us examine the mechanical energy of the block–spring system illustrated in Figure 13.6. Because the surface is frictionless, we expect the total mechanical energy to be constant, as was shown in Chapter 8. We can use Equation 13.7 to ex-
press the kinetic energy as

\[ K = \frac{1}{2} mv^2 = \frac{1}{2} ma^2 A^2 \sin^2(\omega t + \phi) \]  

(13.20)

The elastic potential energy stored in the spring for any elongation \( x \) is given by \( \frac{1}{2} kx^2 \) (see Eq. 8.4). Using Equation 13.3, we obtain

\[ U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) \]  

(13.21)

We see that \( K \) and \( U \) are always positive quantities. Because we can express the total mechanical energy of the simple harmonic oscillator as

\[ E = K + U = \frac{1}{2} kA^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] \]

From the identity \( \sin^2 \theta + \cos^2 \theta = 1 \), we see that the quantity in square brackets is unity. Therefore, this equation reduces to

\[ E = \frac{1}{2} kA^2 \]  

(13.22)

That is, the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude. Note that \( U \) is small when \( K \) is large, and vice versa, because the sum must be constant.

In fact, the total mechanical energy is equal to the maximum potential energy stored in the spring when \( x = \pm A \) because \( v = 0 \) at these points and thus there is no kinetic energy. At the equilibrium position, where \( U = 0 \) because \( x = 0 \), the total energy, all in the form of kinetic energy, is again \( \frac{1}{2} kA^2 \). That is,

\[ E = \frac{1}{2} mv^2_{\text{max}} = \frac{1}{2} m a^2 A^2 = \frac{1}{2} m \frac{k}{m} A^2 = \frac{1}{2} kA^2 \quad \text{(at } x = 0) \]

Plots of the kinetic and potential energies versus time appear in Figure 13.9a, where we have taken \( \phi = 0 \). As already mentioned, both \( K \) and \( U \) are always positive, and at all times their sum is a constant equal to \( \frac{1}{2} kA^2 \), the total energy of the system. The variations of \( K \) and \( U \) with the displacement \( x \) of the block are plotted.

\[ K = \frac{1}{2} m v^2 \]

\[ U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) \]

Figure 13.9  
(a) Kinetic energy and potential energy versus time for a simple harmonic oscillator with \( \phi = 0 \). (b) Kinetic energy and potential energy versus displacement for a simple harmonic oscillator. In either plot, note that \( K + U = \) constant.
in Figure 13.9b. Energy is continuously being transformed between potential energy stored in the spring and kinetic energy of the block.

Figure 13.10 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block–spring system for one full period of the motion. Most of the ideas discussed so far are incorporated in this important figure. Study it carefully.

Finally, we can use the principle of conservation of energy to obtain the velocity for an arbitrary displacement by expressing the total energy at some arbitrary position \( x \) as

\[
E = K + U = \frac{1}{2} mv^2 + \frac{1}{2} kx^2 = \frac{1}{2} kA^2
\]

\[
v = \pm \sqrt{\frac{k}{m} (A^2 - x^2)} = \pm \omega \sqrt{A^2 - x^2}
\]  

(13.23)

When we check Equation 13.23 to see whether it agrees with known cases, we find that it substantiates the fact that the speed is a maximum at \( x = 0 \) and is zero at the turning points \( x = \pm A \).

<table>
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<tr>
<td>T</td>
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<td>0</td>
<td>-( \omega^2 A )</td>
<td>0</td>
<td>( \frac{1}{2} kA^2 )</td>
</tr>
</tbody>
</table>

Figure 13.10 Simple harmonic motion for a block–spring system and its relationship to the motion of a simple pendulum. The parameters in the table refer to the block–spring system, assuming that \( x = A \) at \( t = 0 \); thus, \( x = A \cos \omega t \) (see Special Case 1).
You may wonder why we are spending so much time studying simple harmonic oscillators. We do so because they are good models of a wide variety of physical phenomena. For example, recall the Lennard–Jones potential discussed in Example 8.11. This complicated function describes the forces holding atoms together. Figure 13.11a shows that, for small displacements from the equilibrium position, the potential energy curve for this function approximates a parabola, which represents the potential energy function for a simple harmonic oscillator. Thus, we can approximate the complex atomic binding forces as tiny springs, as depicted in Figure 13.11b.

The ideas presented in this chapter apply not only to block–spring systems and atoms, but also to a wide range of situations that include bungee jumping, tuning in a television station, and viewing the light emitted by a laser. You will see more examples of simple harmonic oscillators as you work through this book.

### Example 13.4 Oscillations on a Horizontal Surface

A 0.500-kg cube connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless track. (a) Calculate the total energy of the system and the maximum speed of the cube if the amplitude of the motion is 3.00 cm.

**Solution** Using Equation 13.22, we obtain

\[ E = K + U = \frac{1}{2} kA^2 = \frac{1}{2} (20.0 \text{ N/m}) (3.00 \times 10^{-2} \text{ m})^2 \]

\[ = 9.00 \times 10^{-3} \text{ J} \]

When the cube is at \( x = 0 \), we know that \( U = 0 \) and \( E = \frac{1}{2} mv_{\text{max}}^2 \); therefore,

\[ \frac{1}{2} mv_{\text{max}}^2 = 9.00 \times 10^{-3} \text{ J} \]

\[ v_{\text{max}} = \sqrt{\frac{18.0 \times 10^{-3} \text{ J}}{0.500 \text{ kg}}} = 0.190 \text{ m/s} \]

(b) What is the velocity of the cube when the displacement is 2.00 cm?

**Solution** We can apply Equation 13.23 directly:

\[ v = \pm \sqrt{\frac{k}{m} (A^2 - x^2)} \]

\[ = \pm \sqrt{\frac{20.0 \text{ N/m}}{0.500 \text{ kg}} [0.030 \text{ m}^2 - (0.020 \text{ m})^2]} \]

\[ = \pm 0.141 \text{ m/s} \]

The positive and negative signs indicate that the cube could be moving to either the right or the left at this instant.

(c) Compute the kinetic and potential energies of the system when the displacement is 2.00 cm.
CHAPTER 13 Oscillatory Motion

The pendulum is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass \( m \) suspended by a light string of length \( L \) that is fixed at the upper end, as shown in Figure 13.12. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle \( \theta \) is small (less than about 10°), the motion is that of a simple harmonic oscillator.

The forces acting on the bob are the force \( T \) exerted by the string and the gravitational force \( mg \). The tangential component of the gravitational force, \( mg \sin \theta \), always acts toward \( \theta = 0 \), opposite the displacement. Therefore, the tangential force is a restoring force, and we can apply Newton’s second law for motion in the tangential direction:

\[
\sum F_i = -mg \sin \theta = m \frac{d^2 s}{dt^2}
\]

where \( s \) is the bob’s displacement measured along the arc and the minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because \( s = L\theta \) (Eq. 10.1a) and \( L \) is constant, this equation reduces to

\[
\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin \theta
\]

The right side is proportional to \( \sin \theta \) rather than to \( \theta \); hence, with \( \sin \theta \) present, we would not expect simple harmonic motion because this expression is not of the form of Equation 13.17. However, if we assume that \( \theta \) is small, we can use the approximation \( \sin \theta \approx \theta \); thus the equation of motion for the simple pen-

![Figure 13.12](image_url)
The pendulum becomes

\[ \frac{d^2 \theta}{dt^2} = -\frac{g}{L} \theta \]  \hspace{1cm} (13.24)

Now we have an expression of the same form as Equation 13.17, and we conclude that the motion for small amplitudes of oscillation is simple harmonic motion. Therefore, \( \theta \) can be written as \( \theta = \theta_{\text{max}} \cos(\omega t + \phi) \), where \( \theta_{\text{max}} \) is the maximum angular displacement and the angular frequency \( \omega \) is

\[ \omega = \sqrt{\frac{g}{L}} \]  \hspace{1cm} (13.25)

The Foucault pendulum at the Franklin Institute in Philadelphia. This type of pendulum was first used by the French physicist Jean Foucault to verify the Earth’s rotation experimentally. As the pendulum swings, the vertical plane in which it oscillates appears to rotate as the bob successively knocks over the indicators arranged in a circle on the floor. In reality, the plane of oscillation is fixed in space, and the Earth rotating beneath the swinging pendulum moves the indicators into position to be knocked down, one after the other.
The period of the motion is

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]  

(13.26)

In other words, the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity. Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and that are at the same location (so that \( g \) is constant) oscillate with the same period. The analogy between the motion of a simple pendulum and that of a block–spring system is illustrated in Figure 13.10.

The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of \( g \). It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of \( g \) can provide information on the location of oil and of other valuable underground resources.

**Quick Quiz 13.5**

A block of mass \( m \) is first allowed to hang from a spring in static equilibrium. It stretches the spring a distance \( L \) beyond the spring’s unstressed length. The block and spring are then set into oscillation. Is the period of this system less than, equal to, or greater than the period of a simple pendulum having a length \( L \) and a bob mass \( m \)?

**Example 13.5** — A Connection Between Length and Time

Christian Huygens (1629–1695), the greatest clockmaker in history, suggested that an international unit of length could be defined as the length of a simple pendulum having a period of exactly 1 s. How much shorter would our length unit be had his suggestion been followed?

**Solution** Solving Equation 13.26 for the length gives

\[ L = \frac{T^2 g}{4\pi^2} = \frac{(1 \text{ s})^2 (9.80 \text{ m/s}^2)}{4\pi^2} = 0.248 \text{ m} \]

Thus, the meter’s length would be slightly less than one-fourth its current length. Note that the number of significant digits depends only on how precisely we know \( g \) because the time has been defined to be exactly 1 s.

**QuickLab**

Firmly hold a ruler so that about half of it is over the edge of your desk. With your other hand, pull down and then release the free end, watching how it vibrates. Now slide the ruler so that only about a quarter of it is free to vibrate. This time when you release it, how does the vibrational period compare with its earlier value? Why?

**Physical Pendulum**

Suppose you balance a wire coat hanger so that the hook is supported by your extended index finger. When you give the hanger a small displacement (with your other hand) and then release it, it oscillates. If a hanging object oscillates about a fixed axis that does not pass through its center of mass and the object cannot be approximated as a point mass, we cannot treat the system as a simple pendulum. In this case the system is called a **physical pendulum.**

Consider a rigid body pivoted at a point \( O \) that is a distance \( d \) from the center of mass (Fig. 13.13). The force of gravity provides a torque about an axis through \( O \), and the magnitude of that torque is \( mgd \sin \theta \), where \( \theta \) is as shown in Figure 13.13. Using the law of motion \( \Sigma \tau = Ia \), where \( I \) is the moment of inertia about
the axis through \( O \), we obtain

\[-mgd \sin \theta = I \frac{d^2 \theta}{dt^2}\]

The minus sign indicates that the torque about \( O \) tends to decrease \( \theta \). That is, the force of gravity produces a restoring torque. Because this equation gives us the angular acceleration \( \frac{d^2 \theta}{dt^2} \) of the pivoted body, we can consider it the equation of motion for the system. If we again assume that \( \theta \) is small, the approximation \( \sin \theta \approx \theta \) is valid, and the equation of motion reduces to

\[ \frac{d^2 \theta}{dt^2} = -\left( \frac{mgd}{I} \right) \theta = -\omega^2 \theta \]

(13.27)

Because this equation is of the same form as Equation 13.17, the motion is simple harmonic motion. That is, the solution of Equation 13.27 is \( \theta = \theta_{\text{max}} \cos(\omega t + \phi) \), where \( \theta_{\text{max}} \) is the maximum angular displacement and

\[ \omega = \sqrt{\frac{mgd}{I}} \]

The period is

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgd}} \]

(13.28)

One can use this result to measure the moment of inertia of a flat rigid body. If the location of the center of mass—and hence the value of \( d \)—are known, the moment of inertia can be obtained by measuring the period. Finally, note that Equation 13.28 reduces to the period of a simple pendulum (Eq. 13.26) when \( I = md^2 \)—that is, when all the mass is concentrated at the center of mass.

**Example 13.6**  A Swinging Rod

A uniform rod of mass \( M \) and length \( L \) is pivoted about one end and oscillates in a vertical plane (Fig. 13.14). Find the period of oscillation if the amplitude of the motion is small.

**Solution**  In Chapter 10 we found that the moment of inertia of a uniform rod about an axis through one end is \( \frac{1}{3}ML^2 \). The distance \( d \) from the pivot to the center of mass is \( L/2 \). Substituting these quantities into Equation 13.28 gives

\[ T = \frac{2\pi}{\sqrt{\frac{\frac{1}{3}ML^2}{Mg \frac{L}{2}}}} = 2\pi \sqrt{\frac{2L}{3g}} \]

**Exercise**  Calculate the period of a meter stick that is pivoted about one end and is oscillating in a vertical plane.

**Answer**  1.64 s.

**Comment**  In one of the Moon landings, an astronaut walking on the Moon’s surface had a belt hanging from his space suit, and the belt oscillated as a physical pendulum. A scientist on the Earth observed this motion on television and used it to estimate the free-fall acceleration on the Moon. How did the scientist make this calculation?

**Figure 13.14**  A rigid rod oscillating about a pivot through one end is a physical pendulum with \( d = L/2 \) and, from Table 10.2, \( I = \frac{1}{3}ML^2 \).
**Figure 13.15** A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support. The body oscillates about the line OP with an amplitude θ_{max}.

**Figure 13.16** The balance wheel of this antique pocket watch is a torsional pendulum and regulates the time-keeping mechanism.

**Torsional Pendulum**

Figure 13.15 shows a rigid body suspended by a wire attached at the top to a fixed support. When the body is twisted through some small angle θ, the twisted wire exerts on the body a restoring torque that is proportional to the angular displacement. That is,

\[ \tau = -\kappa \theta \]

where \(\kappa\) (kappa) is called the torsion constant of the support wire. The value of \(\kappa\) can be obtained by applying a known torque to twist the wire through a measurable angle \(\theta\). Applying Newton’s second law for rotational motion, we find

\[ \tau = -\kappa \theta = I \frac{d^2\theta}{dt^2} \]

\[ \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I} \theta \]

(13.29)

Again, this is the equation of motion for a simple harmonic oscillator, with \(\omega = \sqrt{\kappa/I}\) and a period

\[ T = 2\pi \sqrt{\frac{I}{\kappa}} \]

(13.30)

This system is called a torsional pendulum. There is no small-angle restriction in this situation as long as the elastic limit of the wire is not exceeded. Figure 13.16 shows the balance wheel of a watch oscillating as a torsional pendulum, energized by the mainspring.

**13.5 COMPARING SIMPLE HARMONIC MOTION WITH UNIFORM CIRCULAR MOTION**

We can better understand and visualize many aspects of simple harmonic motion by studying its relationship to uniform circular motion. Figure 13.17 is an overhead view of an experimental arrangement that shows this relationship. A ball is attached to the rim of a turntable of radius \(A\), which is illuminated from the side by a lamp. The ball casts a shadow on a screen. We find that as the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.
Consider a particle located at point $P$ on the circumference of a circle of radius $A$, as shown in Figure 13.18a, with the line $OP$ making an angle $\phi$ with the $x$ axis at $t = 0$. We call this circle a reference circle for comparing simple harmonic motion and uniform circular motion, and we take the position of $P$ at $t = 0$ as our reference position. If the particle moves along the circle with constant angular speed $\omega$ until $OP$ makes an angle $\theta$ with the $x$ axis, as illustrated in Figure 13.18b, then at some time $t > 0$, the angle between $OP$ and the $x$ axis is $\theta = \omega t + \phi$. As the particle moves along the circle, the projection of $P$ on the $x$ axis, labeled point $Q$, moves back and forth along the $x$ axis, between the limits $x = \pm A$.

Note that points $P$ and $Q$ always have the same $x$ coordinate. From the right triangle $OPQ$, we see that this $x$ coordinate is

$$x = A \cos(\omega t + \phi) \quad (13.31)$$

This expression shows that the point $Q$ moves with simple harmonic motion along the $x$ axis. Therefore, we conclude that

simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

We can make a similar argument by noting from Figure 13.18b that the projection of $P$ along the $y$ axis also exhibits simple harmonic motion. Therefore, uniform circular motion can be considered a combination of two simple harmonic motions, one along the $x$ axis and one along the $y$ axis, with the two differing in phase by $90^\circ$.

This geometric interpretation shows that the time for one complete revolution of the point $P$ on the reference circle is equal to the period of motion $T$ for simple harmonic motion between $x = \pm A$. That is, the angular speed $\omega$ of $P$ is the same as the angular frequency $\omega$ of simple harmonic motion along the $x$ axis (this is why we use the same symbol). The phase constant $\phi$ for simple harmonic motion corresponds to the initial angle that $OP$ makes with the $x$ axis. The radius $A$ of the reference circle equals the amplitude of the simple harmonic motion.

**Figure 13.17** An experimental setup for demonstrating the connection between simple harmonic motion and uniform circular motion. As the ball rotates on the turntable with constant angular speed, its shadow on the screen moves back and forth in simple harmonic motion.

**Figure 13.18** Relationship between the uniform circular motion of a point $P$ and the simple harmonic motion of a point $Q$. A particle at $P$ moves in a circle of radius $A$ with constant angular speed $\omega$. (a) A reference circle showing the position of $P$ at $t = 0$. (b) The $x$ coordinates of points $P$ and $Q$ are equal and vary in time as $x = A \cos(\omega t + \phi)$. (c) The $x$ component of the velocity of $P$ equals the velocity of $Q$. (d) The $x$ component of the acceleration of $P$ equals the acceleration of $Q$. 
Because the relationship between linear and angular speed for circular motion is \( v = r\omega \) (see Eq. 10.10), the particle moving on the reference circle of radius \( A \) has a velocity of magnitude \( \omega A \). From the geometry in Figure 13.18c, we see that the \( x \) component of this velocity is \(-\omega A \sin(\omega t + \phi)\). By definition, the point \( Q \) has a velocity given by \( \frac{dx}{dt} \). Differentiating Equation 13.31 with respect to time, we find that the velocity of \( Q \) is the same as the \( x \) component of the velocity of \( P \).

The acceleration of \( P \) on the reference circle is directed radially inward toward \( O \) and has a magnitude \( \frac{v^2}{A} \). From the geometry in Figure 13.18d, we see that the \( x \) component of this acceleration is \( \frac{v^2}{A} \sin(\omega t + \phi) \). This value is also the acceleration of the projected point \( Q \) along the \( x \) axis, as you can verify by taking the second derivative of Equation 13.31.

**Example 13.7**  
Circular Motion with Constant Angular Speed

A particle rotates counterclockwise in a circle of radius 3.00 m with a constant angular speed of 8.00 rad/s. At \( t = 0 \), the particle has an \( x \) coordinate of 2.00 m and is moving to the right. (a) Determine the \( x \) coordinate as a function of time.

**Solution**  
Because the amplitude of the particle’s motion equals the radius of the circle and \( \omega = 8.00 \text{ rad/s} \), we have

\[
x = A \cos(\omega t + \phi) = (3.00 \text{ m}) \cos(8.00t + \phi)
\]

We can evaluate \( \phi \) by using the initial condition that \( x = 2.00 \text{ m} \) at \( t = 0 \):

\[
2.00 \text{ m} = (3.00 \text{ m}) \cos(0 + \phi)
\]

\[
\phi = \cos^{-1} \left( \frac{2.00 \text{ m}}{3.00 \text{ m}} \right)
\]

If we were to take our answer as \( \phi = 48.2^\circ \), then the coordinate \( x = (3.00 \text{ m}) \cos(8.00t + 48.2^\circ) \) would be decreasing at time \( t = 0 \) (that is, moving to the left). Because our particle is first moving to the right, we must choose \( \phi = -48.2^\circ = -0.841 \text{ rad} \). The \( x \) coordinate as a function of time is then

\[
x = (3.00 \text{ m}) \cos(8.00t - 0.841)
\]

Note that \( \phi \) in the cosine function must be in radians.

(b) Find the \( x \) components of the particle’s velocity and acceleration at any time \( t \).

**Solution**

\[
v_x = \frac{dx}{dt} = -(3.00 \text{ m})(8.00 \text{ rad/s}) \sin(8.00t - 0.841)
\]

\[
= -24.0 \text{ m/s} \sin(8.00t - 0.841)
\]

\[
a_x = \frac{dv_x}{dt} = -(24.0 \text{ m/s})(8.00 \text{ rad/s}) \cos(8.00t - 0.841)
\]

\[
= -(192 \text{ m/s}^2) \cos(8.00t - 0.841)
\]

From these results, we conclude that \( v_{\text{max}} = 24.0 \text{ m/s} \) and that \( a_{\text{max}} = 192 \text{ m/s}^2 \). Note that these values also equal the tangential speed \( \omega A \) and the centripetal acceleration \( \omega^2 A \).

**Optional Section**

**13.6 DAMPED OSCILLATIONS**

The oscillatory motions we have considered so far have been for ideal systems—that is, systems that oscillate indefinitely under the action of a linear restoring force. In many real systems, dissipative forces, such as friction, retard the motion. Consequently, the mechanical energy of the system diminishes in time, and the motion is said to be damped.

One common type of retarding force is the one discussed in Section 6.4, where the force is proportional to the speed of the moving object and acts in the direction opposite the motion. This retarding force is often observed when an object moves through air, for instance. Because the retarding force can be expressed as \( \mathbf{R} = -b \mathbf{v} \) (where \( b \) is a constant called the damping coefficient) and the restoring
force of the system is \(-kx\), we can write Newton’s second law as

\[
\sum F_x = -kx - bv = ma_x
\]

\[
-kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}
\]  

(13.32)

The solution of this equation requires mathematics that may not be familiar to you yet; we simply state it here without proof. When the retarding force is small compared with the maximum restoring force—that is, when \(b\) is small—the solution to Equation 13.32 is

\[
x = Ae^{-\frac{b}{2m}t} \cos(\omega t + \phi)
\]

(13.33)

where the angular frequency of oscillation is

\[
\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}
\]

(13.34)

This result can be verified by substituting Equation 13.33 into Equation 13.32.

Figure 13.19a shows the displacement as a function of time for an object oscillating in the presence of a retarding force, and Figure 13.19b depicts one such system: a block attached to a spring and submersed in a viscous liquid. We see that when the retarding force is much smaller than the restoring force, the oscillatory character of the motion is preserved but the amplitude decreases in time, with the result that the motion ultimately ceases. Any system that behaves in this way is known as a damped oscillator. The dashed blue lines in Figure 13.19a, which define the envelope of the oscillatory curve, represent the exponential factor in Equation 13.33. This envelope shows that the amplitude decays exponentially with time. For motion with a given spring constant and block mass, the oscillations dampen more rapidly as the maximum value of the retarding force approaches the maximum value of the restoring force.

It is convenient to express the angular frequency of a damped oscillator in the form

\[
\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}
\]

where \(\omega_0 = \sqrt{k/m}\) represents the angular frequency in the absence of a retarding force (the undamped oscillator) and is called the natural frequency of the system. When the magnitude of the maximum retarding force \(R_{\text{max}} = bv_{\text{max}} < kA\), the system is said to be underdamped. As the value of \(R\) approaches \(kA\), the amplitudes of the oscillations decrease more and more rapidly. This motion is represented by the blue curve in Figure 13.20. When \(b\) reaches a critical value \(b_c\) such that \(b_c/2m = \omega_0\), the system does not oscillate and is said to be critically damped. In this case the system, once released from rest at some nonequilibrium position, returns to equilibrium and then stays there. The graph of displacement versus time for this case is the red curve in Figure 13.20.

If the medium is so viscous that the retarding force is greater than the restoring force—that is, if \(R_{\text{max}} = bv_{\text{max}} > kA\) and \(b'/2m > \omega_0\)—the system is over-damped. Again, the displaced system, when free to move, does not oscillate but simply returns to its equilibrium position. As the damping increases, the time it takes the system to approach equilibrium also increases, as indicated by the black curve in Figure 13.20.

In any case in which friction is present, whether the system is overdamped or underdamped, the energy of the oscillator eventually falls to zero. The lost mechanical energy dissipates into internal energy in the retarding medium.
An automotive suspension system consists of a combination of springs and shock absorbers, as shown in Figure 13.21. If you were an automotive engineer, would you design a suspension system that was underdamped, critically damped, or overdamped? Discuss each case.

Optional Section

FORCED OSCILLATIONS

It is possible to compensate for energy loss in a damped system by applying an external force that does positive work on the system. At any instant, energy can be put into the system by an applied force that acts in the direction of motion of the oscillator. For example, a child on a swing can be kept in motion by appropriately timed pushes. The amplitude of motion remains constant if the energy input per cycle exactly equals the energy lost as a result of damping. Any motion of this type is called forced oscillation.

A common example of a forced oscillator is a damped oscillator driven by an external force that varies periodically, such as \( F = F_{\text{ext}} \cos \omega t \), where \( \omega \) is the angular frequency of the periodic force and \( F_{\text{ext}} \) is a constant. Adding this driving force to the left side of Equation 13.32 gives

\[
F_{\text{ext}} \cos \omega t - k x - b \frac{dx}{dt} = m \frac{d^2x}{dt^2} \tag{13.35}
\]

(As earlier, we present the solution of this equation without proof.) After a sufficiently long period of time, when the energy input per cycle equals the energy lost per cycle, a steady-state condition is reached in which the oscillations proceed with constant amplitude. At this time, when the system is in a steady state, the solution of Equation 13.35 is

\[
x = A \cos(\omega t + \phi) \tag{13.36}
\]
where

\[ A = \frac{F_{\text{ext}}/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + \left(\frac{b\omega}{m}\right)^2}} \]  

(13.37)

and where \( \omega_0 = \sqrt{k/m} \) is the angular frequency of the undamped oscillator \((b = 0)\).

One could argue that in steady state the oscillator must physically have the same frequency as the driving force, and thus the solution given by Equation 13.36 is expected. In fact, when this solution is substituted into Equation 13.35, one finds that it is indeed a solution, provided the amplitude is given by Equation 13.37.

Equation 13.37 shows that, because an external force is driving it, the motion of the forced oscillator is not damped. The external agent provides the necessary energy to overcome the losses due to the retarding force. Note that the system oscillates at the angular frequency \( \omega \) of the driving force. For small damping, the amplitude becomes very large when the frequency of the driving force is near the natural frequency of oscillation. The dramatic increase in amplitude near the natural frequency \( \omega_0 \) is called resonance, and for this reason \( \omega_0 \) is sometimes called the resonance frequency of the system.

The reason for large-amplitude oscillations at the resonance frequency is that energy is being transferred to the system under the most favorable conditions. We can better understand this by taking the first time derivative of \( x \) in Equation 13.36, which gives an expression for the velocity of the oscillator. We find that \( v \) is proportional to \( \sin(\omega t + \phi) \). When the applied force \( F \) is in phase with the velocity, the rate at which work is done on the oscillator by \( F \) equals the dot product \( F \cdot v \). Remember that “rate at which work is done” is the definition of power. Because the product \( F \cdot v \) is a maximum when \( F \) and \( v \) are in phase, we conclude that at resonance the applied force is in phase with the velocity and that the power transferred to the oscillator is a maximum.

Figure 13.22 is a graph of amplitude as a function of frequency for a forced oscillator with and without damping. Note that the amplitude increases with decreasing damping \((b \to 0)\) and that the resonance curve broadens as the damping increases. Under steady-state conditions and at any driving frequency, the energy transferred into the system equals the energy lost because of the damping force; hence, the average total energy of the oscillator remains constant. In the absence of a damping force \((b = 0)\), we see from Equation 13.37 that the steady-state amplitude approaches infinity as \( \omega \to \omega_0 \). In other words, if there are no losses in the system and if we continue to drive an initially motionless oscillator with a periodic force that is in phase with the velocity, the amplitude of motion builds without limit (see the red curve in Fig. 13.22). This limitless building does not occur in practice because some damping is always present.

The behavior of a driven oscillating system after the driving force is removed depends on \( b \) and on how close \( \omega \) was to \( \omega_0 \). This behavior is sometimes quantified by a parameter called the quality factor \( Q \). The closer a system is to being undamped, the greater its \( Q \). The amplitude of oscillation drops by a factor of \( e (\approx 2.718 \ldots) \) in \( Q/\pi \) cycles.

Later in this book we shall see that resonance appears in other areas of physics. For example, certain electrical circuits have natural frequencies. A bridge has natural frequencies that can be set into resonance by an appropriate driving force. A dramatic example of such resonance occurred in 1940, when the Tacoma Narrows Bridge in the state of Washington was destroyed by resonant vibrations. Although the winds were not particularly strong on that occasion, the bridge ultimately collapsed (Fig. 13.23) because the bridge design had no built-in safety features.

**Figure 13.22** Graph of amplitude versus frequency for a damped oscillator when a periodic driving force is present. When the frequency of the driving force equals the natural frequency \( \omega_0 \), resonance occurs. Note that the shape of the resonance curve depends on the size of the damping coefficient \( b \).

**QuickLab**

Tie several objects to strings and suspend them from a horizontal string, as illustrated in the figure. Make two of the hanging strings approximately the same length. If one of this pair, such as \( P \), is set into sideways motion, all the others begin to oscillate. But \( Q \), whose length is the same as that of \( P \), oscillates with the greatest amplitude. Must all the masses have the same value?
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Many other examples of resonant vibrations can be cited. A resonant vibration that you may have experienced is the “singing” of telephone wires in the wind. Machines often break if one vibrating part is at resonance with some other moving part. Soldiers marching in cadence across a bridge have been known to set up resonant vibrations in the structure and thereby cause it to collapse. Whenever any real physical system is driven near its resonance frequency, you can expect oscillations of very large amplitudes.

Summary

When the acceleration of an object is proportional to its displacement from some equilibrium position and is in the direction opposite the displacement, the object moves with simple harmonic motion. The position $x$ of a simple harmonic oscillator varies periodically in time according to the expression

$$x = A \cos(\omega t + \phi) \quad (13.3)$$

where $A$ is the amplitude of the motion, $\omega$ is the angular frequency, and $\phi$ is the phase constant. The value of $\phi$ depends on the initial position and initial velocity of the oscillator. You should be able to use this formula to describe the motion of an object undergoing simple harmonic motion.

The time $T$ needed for one complete oscillation is defined as the period of the motion:

$$T = \frac{2\pi}{\omega} \quad (13.4)$$

The inverse of the period is the frequency of the motion, which equals the number of oscillations per second.

The velocity and acceleration of a simple harmonic oscillator are

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi) \quad (13.7)$$

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) \quad (13.8)$$

$$v = \pm \omega \sqrt{A^2 - x^2} \quad (13.23)$$

Figure 13.23  (a) In 1940 turbulent winds set up torsional vibrations in the Tacoma Narrows Bridge, causing it to oscillate at a frequency near one of the natural frequencies of the bridge structure. (b) Once established, this resonance condition led to the bridge’s collapse.
Thus, the maximum speed is $\omega A$, and the maximum acceleration is $\omega^2 A$. The speed is zero when the oscillator is at its turning points, $x = \pm A$, and is a maximum when the oscillator is at the equilibrium position $x = 0$. The magnitude of the acceleration is a maximum at the turning points and zero at the equilibrium position. You should be able to find the velocity and acceleration of an oscillating object at any time if you know the amplitude, angular frequency, and phase constant.

A block–spring system moves in simple harmonic motion on a frictionless surface, with a period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad (13.18)$$

The kinetic energy and potential energy for a simple harmonic oscillator vary with time and are given by

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \quad (13.20)$$

$$U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \cos^2(\omega t + \phi) \quad (13.21)$$

These three formulas allow you to analyze a wide variety of situations involving oscillations. Be sure you recognize how the mass of the block and the spring constant of the spring enter into the calculations.

The total energy of a simple harmonic oscillator is a constant of the motion and is given by

$$E = \frac{1}{2} kA^2 \quad (13.22)$$

The potential energy of the oscillator is a maximum when the oscillator is at its turning points and is zero when the oscillator is at the equilibrium position. The kinetic energy is zero at the turning points and a maximum at the equilibrium position. You should be able to determine the division of energy between potential and kinetic forms at any time $t$.

A simple pendulum of length $L$ moves in simple harmonic motion. For small angular displacements from the vertical, its period is

$$T = 2\pi \sqrt{\frac{L}{g}} \quad (13.26)$$

For small angular displacements from the vertical, a physical pendulum moves in simple harmonic motion about a pivot that does not go through the center of mass. The period of this motion is

$$T = 2\pi \sqrt{\frac{l}{mgd}} \quad (13.28)$$

where $l$ is the moment of inertia about an axis through the pivot and $d$ is the distance from the pivot to the center of mass. You should be able to distinguish when to use the simple-pendulum formula and when the system must be considered a physical pendulum.

Uniform circular motion can be considered a combination of two simple harmonic motions, one along the $x$ axis and the other along the $y$ axis, with the two differing in phase by $90^\circ$.

**Questions**

1. Is a bouncing ball an example of simple harmonic motion?
   Is the daily movement of a student from home to school and back simple harmonic motion? Why or why not?

2. If the coordinate of a particle varies as $x = -A \cos \omega t$, what is the phase constant in Equation 13.3? At what position does the particle begin its motion?