Assignment 4: Quantum Field Theory Due Date: 26 Feb. 4 pm

1. Using the definition of the Fourier transform of $\hat{x}_k$ and the decomposition of $\hat{x}_k$ in terms of the creation and annihilation operators, derive the following mode expansion,

$$\hat{x}_j = \sqrt{\frac{\hbar}{2m\omega N}} \sum_k \left( \hat{a}_k e^{+ikja} + \hat{a}_k^\dagger e^{-ikja} \right)$$  \hspace{1cm} (1)

2. Show that $[\hat{a}_k, \hat{a}_k^\dagger] = 1$. Your starting point is the position-momentum uncertainty.

3. Consider the two-particle momentum state $|pq\rangle$.

   (a) Show that

   $$\langle p'q'|pq\rangle = \delta^{(3)}(p' - q) \delta^{(3)}(q' - p) \pm \delta^{(3)}(p' - p) \delta^{(3)}(q' - q).$$  \hspace{1cm} (2)

   (b) Using this result show that

   $$\frac{1}{\sqrt{2}} \int d^3p' d^3q' \phi_{p'}(x) \phi_{q'}(y) \langle p'q'|pq\rangle = \frac{1}{\sqrt{2}} \left( \phi_{q}(x)\phi_{p}(y) \pm \phi_{p}(x) \phi_{q}(y) \right)$$  \hspace{1cm} (3)

   where $\phi_{p}(x) = \langle x|p \rangle$.

4. Given the Hamiltonian for $N$ coupled oscillators on a ring,

   $$\hat{H} = \sum_j \left( \frac{p_j^2}{2m} + \frac{1}{2} K (\hat{x}_{j+1}^2 - \hat{x}_j^2) \right),$$  \hspace{1cm} (4)

   derive the k space Hamiltonian. Show the complete working at all steps.

5. For a 3D harmonic oscillator the Hamiltonian is

   $$\hat{H} = \hbar\omega \sum_{i=1,2,3} \left( \hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right).$$  \hspace{1cm} (5)

   With the transformations,

   $$\hat{b}_3^\dagger = \hat{a}_3^\dagger$$  \hspace{1cm} (6)

   $$\hat{b}_1^\dagger = -\frac{1}{\sqrt{2}} (\hat{a}_1^\dagger + i \hat{a}_2^\dagger)$$  \hspace{1cm} (7)

   $$\hat{b}_2^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_1^\dagger - i \hat{a}_2^\dagger)$$  \hspace{1cm} (8)

   show that

   $$\hat{H} = \hbar\omega \sum_{i=1,2,3} \left( \hat{b}_i^\dagger \hat{b}_i \right).$$  \hspace{1cm} (9)

6. Derive the commutation relationship for the fermionic field operators (in space).