Assignment 8: Quantum Field Theory
Due Date: 10 April. 10 am

1. Show a neat derivation of the Heaviside function expanded as an integral,
\[ \Theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dz \frac{e^{-itz}}{z + i0^+}. \]
[10 marks]

2. Is
\[ \Delta(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0^+} e^{-ip(x-y)} \]
the Green function for the Klein-Gordon operator? Show your working. [5 marks]

3. In class, we derived the following expression for the free propagation for scalar fields:
\[ \Delta(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{(p^0)^2 - E_p^2} \]
Replace \( E_p \) with \( E_p - i0^+ \), where \( 0^+ \) is an infinitesimal positive number. Compute the integral given above using the rules of contour integration, verifying in the process, that
\[ \Delta(x, y) = \int \frac{d^3 p}{(2\pi)^3} \left( \Theta(x^0 - y^0) e^{-ip \cdot x} + \Theta(y^0 - x^0) e^{+ip \cdot x} \right). \]
What is the role of the minute term \( i0^+ \)? [15 marks]

4. (a) For a massive scalar field, the Lagrangian density is:
\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi(x) \right)^2 - \frac{1}{2} m^2 \left( \phi(x) \right)^2. \]
Express the action \( S = \int d^4 x \ L \) in the momentum space and comment how \( S \) depends on the momentum representation of the Feynmann propagator for the scalar field. [10 marks]

(b) Now let’s discretize the scalar field by positioning it on an equally spaced chain in one dimension. The discretization is achieved by:
\[ \phi_j(t) = \frac{1}{\sqrt{L}} \sum_p \int d\omega \frac{\sim}{2\pi} \phi_p e^{-i(\omega t - pja)} \]
where \( L = Na \) is the length of the chain, \( j \) is the spatial index and \( \tilde{\phi}_p(\omega) \) is the Fourier transform of \( \phi_j(t) \). Using

\[
\frac{1}{L} \sum_j e^{+i(p+q)ja} = \delta^{(1)}(p + q)
\]

and the fact that we have dealing with the \((1 + 1)\) Minkonski space, derive the action \( S \) in momentum space.

Using results from the previous question, state the Green Function for this one-dimensional chain. The excitations of such a field are called phonons.

[15 marks]

5. Show that only if \( \hat{H}_{11}(t) \) is self-commuting at all times, does

\[
\hat{U}(t_2, t_1) = e^{-i \int_{t_1}^{t_2} \hat{H}_{11}(\tau) \, d\tau}
\]

represent a solution of

\[
i \frac{d}{dt_2} \hat{U}(t_2, t_1) = \hat{H}_{11} \hat{U}(t_2, t_1).
\]

[10 marks]