Dynamics of water discharge using Physlogger*

Umar Hassan, Amrozia Shaheen and Muhammad Sabieh Anwar
LUMS Syed Baber Ali School of Science and Engineering

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This experiment demonstrates the dynamics of water flowing out of a tank. Using the rate change of mass for an emptying cylinder, we investigate the application of Bernoulli's equation and the resulting Torricelli's theorem. We also observe the effects of constriction on the parcel of water flowing out of the tank. Students will investigate fluid dynamics, pressure and will relish how a phenomena as simple as water flowing out from a tank can lead to rich dynamics that can be explored mathematically.

KEYWORDS

Pressure · viscosity · Bernoulli's equation · Torricelli's law · Laminar Flow · Continuity Equation

1 Conceptual Objectives

In this experiment, we will,

1. understand and apply Bernoulli's equation,
2. understand Torricelli's law,
3. understand the continuity equation,
4. learn how to numerically differentiate data, and
5. make plots of variables derived from directly measured quantities.

2 Theoretical background

A fluid is a collection of molecules held together by weak cohesive forces. Usually liquids and gases are termed as fluids because they deform in response to external forces. Some general properties of fluid flow are summarized here.

1. **Steady or non-steady:** The flow of a fluid is described by pressure, density and flow velocity at every point of the fluid. If these variables are constant in time then the flow is steady.
2. **Compressible or incompressible:** If the density of a fluid remains constant and does not depend on x, y, z and t, then the flow is incompressible.
3. **Viscous or non-viscous:** Viscosity is the resistance towards flow. When a fluid flows such that there is no energy dissipation, then it is non-viscous flow. Such a flow is really an idealization.
4. **Rotational or irrotational:** If any element of the fluid does not rotate about an axis through the center of mass of the element, then the flow is irrotational.

2.1 Pressure inside a fluid

Consider a small segment of the fluid of density ρ at a distance y above some reference level as shown in Figure 1(a). This segment is a thin disk with thickness dy and area A, as illustrated in part (b) of the diagram. The mass of the element is $dm = \rho dV = \rho A\, dy$ and its weight $W = (dm)g = \rho g A\, dy$. Since there is no acceleration the net vertical force is zero,

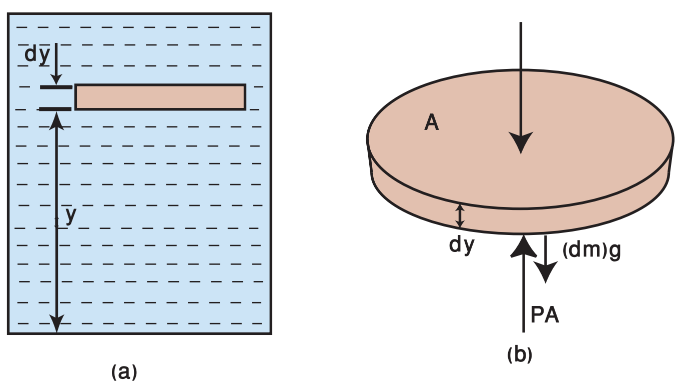


Figure 1: A static fluid. (a) Small element at rest inside the fluid, (b) forces acting on a small element.

$$\Sigma F_y = PA - (P + dP)A - \rho g A\, dy = 0, \quad (1)$$

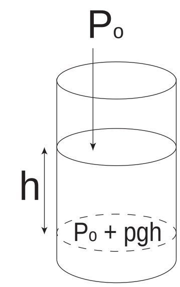
yielding,

$$\frac{dP}{dy} = -\rho g. \tag{2}$$

This equation describes the variation of pressure with elevation above some reference level.

As the height increases (dy positive), the pressure decreases (dP negative). For an incompressible and homogeneous liquid with difference in height, the pressure difference is found by integrating the Equation (2)

$$P_2 - P_1 = -\rho g(y_2 - y_1), \tag{3}$$



and if the liquid has a free surface exposed to the atmospheric pressure P_o, then,

$$\begin{aligned} P_o - P &= -\rho g(y_2 - y_1), \\ P &= P_o + \rho g h, \end{aligned} \tag{4}$$

where, $y_2 - y_1 = h$. This shows that the pressure in a liquid increases with depth but would be same at all those points that are on the same level.

2.2 Bernoulli's equation

In our discussion on pressure, we have seen how pressure depends on the weight of the fluid above a level. However, pressure will also change with speed and elevation. You must have noticed how an object could be pulled into the wake of a fast-moving train; or by narrowing the hose of a water pipe, the stream of water can go further.

Q 1. Why do hordes of birds fly in a characteristic V-shaped pattern?

When a fluid moves through a region in which either the speed of the fluid or elevation above the earth's surface changes, the impact is that the pressure in the fluid changes. The relationship between fluid speed, pressure and elevation was first derived by Daniel Bernoulli in 1738. Bernoulli's equation, a fundamental relation in fluid mechanics is derivable from basic laws of Newtonian mechanics, as well as from the work-energy principle which stems from the conservation of energy.

Consider a steady, incompressible and nonviscous flow of a fluid through a pipeline from the position shown in Figure 2(a) to (b). The portion at the left has a cross sectional area A_1 and at an elevation y_1 from some reference level. A mass of fluid Δm gradually rises and after time Δt, it moves to the right end with cross sectional A_2, at an elevation y_2.

According to the work-energy theorem, the work done by the resultant force that acts on a system is equal to the change in kinetic energy. Assuming that there is no viscous force, the only forces that do work on the system are the pressure forces and the force of gravity. The net work done on the system by all the forces is,

$$W = P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - (\Delta m)\, g(y_2 - y_1). \tag{5}$$

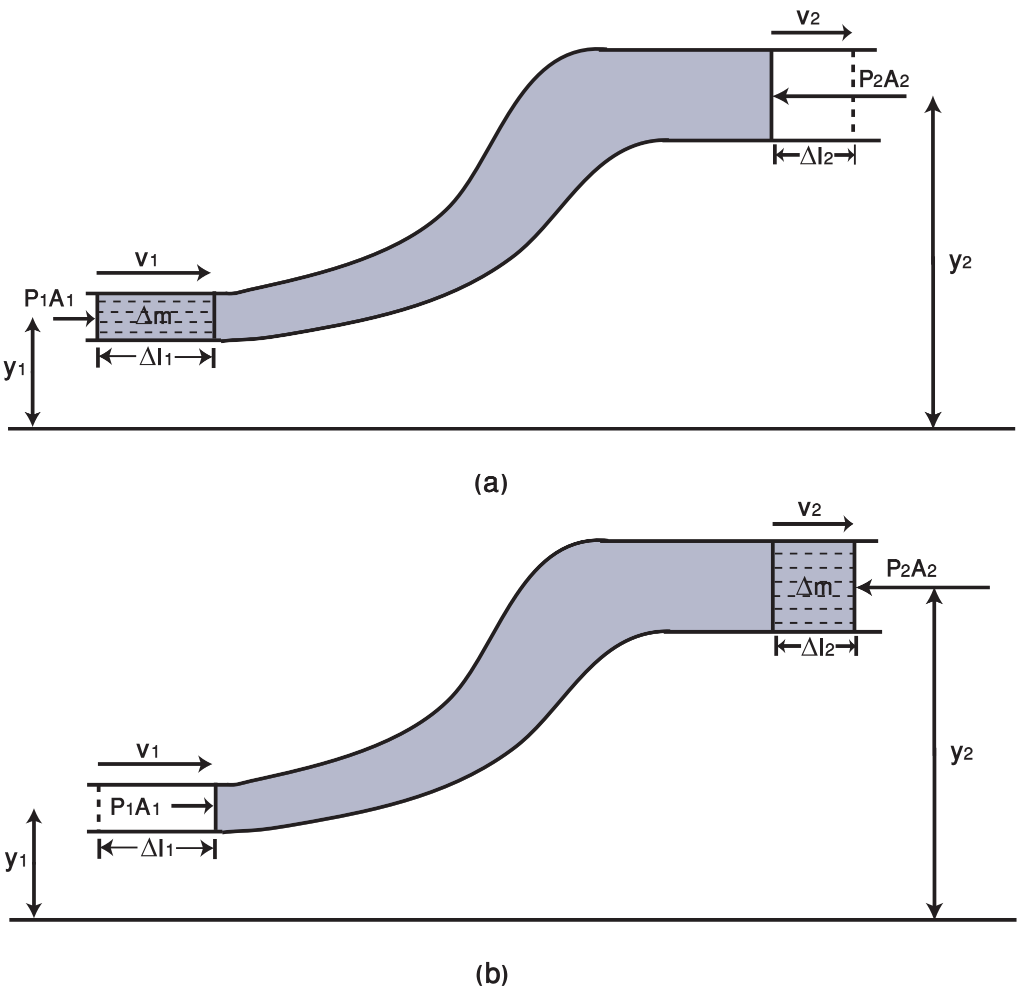


Figure 2: A fluid is flowing through a pipe from position (a) to (b). The net effect is the transfer of the element (Δm) from the left to the right end. We calculate the work done in this transfer process.

This is the work done as the mass Δm displaces from (a) to (b). The pressure force $P_2 A_2 \Delta l_2$ bears a negative sign because its direction is opposite to the horizontal displacement Δl_2. The gravitational force is also negative because it acts in a direction opposite to the vertical displacement. As $A_1 \Delta l_1 = A_2 \Delta l_2$ is the volume of the fluid (ΔV) displaced, we can replace this $\Delta m/\rho$. The change in kinetic energy, therefore is,

$$\begin{aligned}
\Delta K &= \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 \\
&= \frac{1}{2}\Delta m (v_2^2 - v_1^2) \\
&= P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 - (\Delta m) g (y_2 - y_1)
\end{aligned} \tag{6}$$

This can be rearranged to give,

$$\frac{1}{2}\Delta m(v_2^2) + P_2 A_2 \Delta l_2 + (\Delta m) g y_2 = \frac{1}{2}\Delta m(v_1^2) + P_1 A_1 \Delta l_1 + (\Delta m) g y_2 \tag{7}$$

Dividing each side by the respective volume of the element $A_2(\Delta l_2) = A_1(\Delta l_1)$,

$$P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 \quad = \quad P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 \, . \tag{8}$$

The above equation is often expressed as,

$$P + \frac{1}{2}\rho v^2 + \rho g y \quad = \quad \text{constant.} \tag{9}$$

This is a statement of Bernoulli's equation.

The relation in Equation (8) can be modified in many different ways depending upon the situation. This leads to interesting corollaries. If the fluid is at rest i.e. $v_2 = v_1 = 0$ then,

$$P_1 + \rho g h_1 = P_2 + \rho g h_2, \tag{10}$$

where the term $(P + \rho g y)$ is called the *static pressure*.

Likewise, if both ends of the pipe are placed at same height then Equation (9) can be re-written as,

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2, \tag{11}$$

showing that high speeds corresponds to low pressures. The term $\frac{1}{2}\rho v^2$ has dimensions of pressure and is called *dynamic pressure*.

2.3 Continuity equation

The equation of continuity for incompressible fluids states that,

$$\rho A v = \text{constant} \tag{12}$$

where v is the velocity and ρ is density of the fluid. This relation is easy to understand. Consider Figure (3) which shows a tapered horizontal pipe. The area at the left end is A_1 and at the right is $A_2 < A_1$. In a unit time Δt, a mass of liquid Δm is transported between the ends. Since the fluid cannot be compressed, we must conserve the mass of fluid transferred, otherwise the liquid will turn denser in some regions and rarer in others. Therefore,

$$\rho A_1 \Delta l_1 = \rho A_2 \Delta l_2, \tag{13}$$

which dividing by Δt yields the equation of continuity, (12).

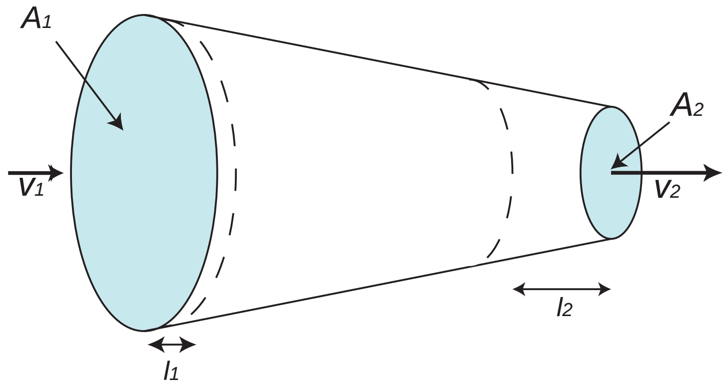
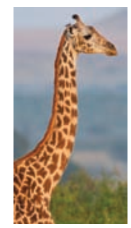


Figure 3: A tapered horizontal pipe. The horizontal velocity vectors are depicted by arrows.



Q 2. A giraffe needs a strong heart because of its long neck. Suppose the difference of height between the aortic valve (the place where the arterial blood comes out of the heart) and the head of a giraffe is 2.50 m, and the artery leading from near the aortic valve to the head has constant cross section all the way to the head. Blood is an incompressible fluid with density $1.0\,\mathrm{g/cm^3}$. Assume the pressure at the head is zero.

(a) What is the minimum required pressure at the aortic valve? Compare this pressure to the peak output pressure of the human heart (1.6×10^4 Pa)?

(b) What would be the effect on the giraffe if the artery diameter narrowed down as it approached the brain?

2.4 Water discharge from a cylinder

A cylinder contains water which flows out from a narrow circular orifice at a fixed height y_2 from the base. [right]right The orifice has a small area A_2 compared to the cross sectional area A_1 of the cylinder. As time progresses, the level of the water $y_1(t)$ in the cylinder descends and water issues out with a speed $v_2(t)$. Let's apply Bernoulli's law to points **1** and **2**. Note that at the orifice, the jet of water is also exposed to the atmospheric pressure P_o. From Bernoulli's principle,

Consider Fig (4)

$$\not{P_o} + \frac{1}{2}\rho v_1^2 + \rho g y_1 = \not{P_o} + \frac{1}{2}\rho v_2^2 + \rho g y_2 \tag{14}$$

Since $A_1 \gg A_2$, $v_1 \approx 0$, leading to

$$\begin{aligned} \tfrac{1}{2}\rho v_2^2 &= \rho g (y_1 - y_2) \\ v_2^2 &= 2gh, \end{aligned} \tag{15}$$

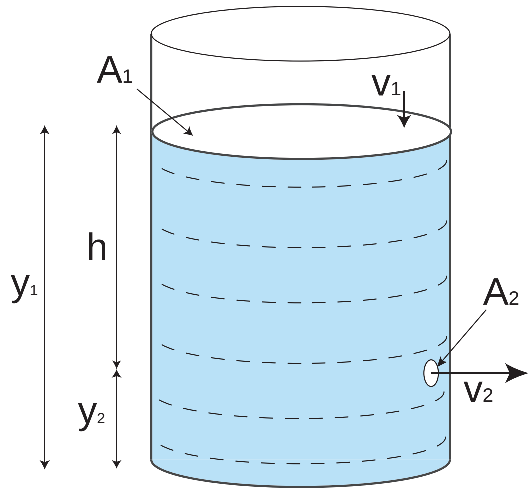


Figure 4: A cylinder with water flowing out from a narrow orifice at a fixed height y_2 from the base.

which shows the relationship between the speed $v_2(t)$ and the instantaneous head $h(t)$ of water *above* the orifice.

2.5 Torricelli's Law

Torricellis Law describes the relationship between the velocity of fluid leaving the cylinder v_2 and the height h of the fluid. This relationship is given in Equation (15). In its simplest form, the speed, v, of a liquid flowing under the force of gravity out of an opening in a tank is proportional to the square root of the vertical distance, h. The speed of efflux is independent of the direction of flow. The theorem is named after Evangelista Torricelli, who formulated it in 1643. Notice that this speed is identical to the speed acquired by a mass falling under gravity through a height h.

In the experiment, you will observe if a linear relationship between v_2^2 and h exists. Furthermore, v_2^2 will in fact be observed to be smaller than $2gh$. The discrepancy will be accounted for by water's viscosity, and the effective narrowing of the orifice.

3 The Experiment

3.1 Preparation

You are provided a graduated cylinder with an orifice at a fixed height y_2 from the base. Place it on the provided Physload balance which is connected to Physlogger. Connect Physlogger to the PC's USB port and open the Physlogger app. Meanwhile place the provided plastic box in the line of the orifice to collect the discharging water. Water runs down through a plastic water guide tube. Complete the assembly as shown in Figure 5.

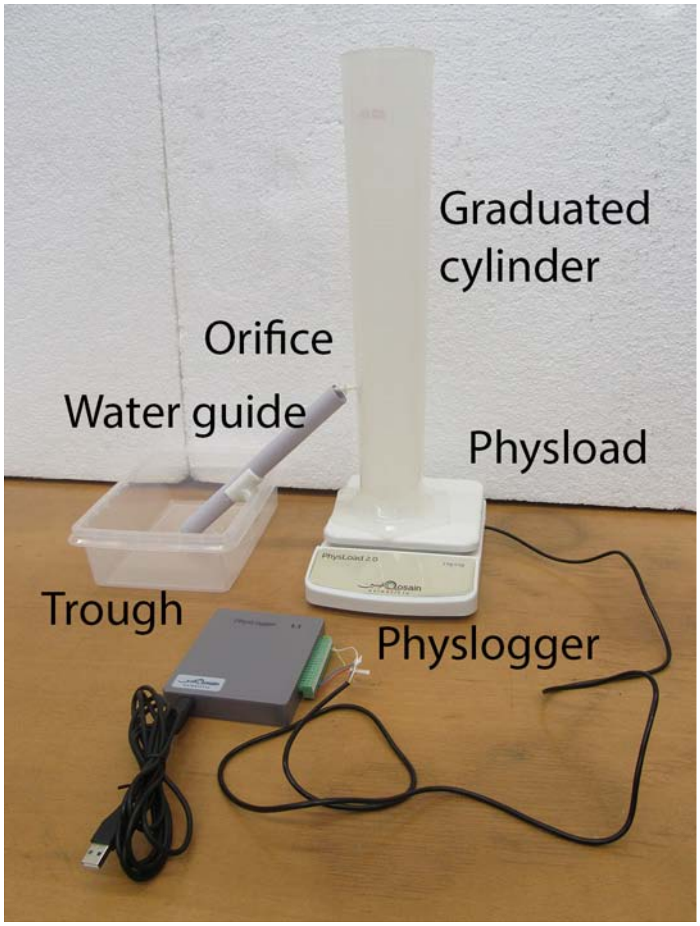


Figure 5: The experimental assembly for observing the rate of discharge from a graduated cylinder.

In the Physlogger app, select the `‘Arduino Mega 2560’` from the drop down list then click `‘connect’`. Choose a sampling rate of 0.2 Hz which means that a sample is acquired after every 5 s. Right click on the area where the graph of the channel displays and click `‘Channel Options’`, then click `‘Type’` and select the device `‘Physload 2.0’`. Once these settings are made, Physlogger will automatically setup the axis and will begin plotting data acquired from Physload.

Now you have mass data being plotted in real-time but by default, the range of the plotted mass is set to 100 g. If you need to change the rang; right click anywhere in the area where the graph is displayed and go to `‘Channel Options’`; then click `‘Range’` and select your desired value. We generally choose 1500 g for a cylinder of volume 1 L.

Once you have set your range you should tare the weight of graduated cylinder. For that purpose, once again, right click in the graphing area and go to `‘Channel Options’`, then click `‘Tare’`. “Taring” means setting the reading of Physload to zero. After you have your desired data acquired and displayed in Physlogger's window, you can export the data using the'export' option provided in the top left corner. You can obviously read through Physlogger's manual for further information [1] or your instructor can provide more details.

Apparatus Parameters	Values
Graduated cylinder diameter	(69 ± 1) mm
Diameter of the orifice	(1.9 ± 0.2) mm

Table 1: Various parameters pertaining the apparatus that will be required for analysis.

3.2 Experimental procedure and analysis

In this experiment, you are required to verify a linear relationship between $v_2^2(t)$ and $h(t)$. The balance will return the mass of water at each time step. Write a computer programme that converts the obtained mass to the mass flow rate. From your programme, you should be able to compute the following quantities.

1. Rate of decrease of mass versus time.
2. Speed v_2 of the discharging water jet versus time.
3. Height h of the descending water level versus time. Note that h is the head above the orifice and the orifice is at a height of 300 mL above the base of the cylinder.
4. Plot v_2 versus h.
5. Plot v_2^2 versus h. Fit this data to a linear relationship. Is it a good fit? Does the data corroborate Torricelli's theorem? Ideally the slope of the v_2^2 verses h graph should have a slope $2g$. Is the slope of your data smaller or greater than $2g$? How do you account for the difference?

References

[1] https://www.physlab.org/facility/physlogger/.